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# CAUCHY-SCHWARZ FUNCTIONS AND CONVEX PARTITIONS IN THE RAY SPACE OF A SUPERTROPICAL QUADRATIC FORM

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Abstract. Rays are classes of an equivalence relation on a module $V$ over a supertropical semiring. They provide a version of convex geometry, supported by a "supertropical trigonometry" and compatible with quasilinearity, in which the CS-ratio takes the role of the Cauchy-Schwarz inequality. CS-functions which emerge from the CS-ratio are a useful tool that helps to understand the variety of quasilinear stars in the ray space $\mathrm{Ray}(V)$. In particular, these functions induce a partition of $\mathrm{Ray}(V)$ into convex sets, and thereby a finer convex analysis which includes the notions of median, minima, glens, and polars.

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## 1. Introduction

Quadratic forms on a free supertropical module, and their bilinear companions, were introduced and classified in [6, 7], and studied further in [4, 8, 9]. These objects establish a version of tropical trigonometry, where the CS-ratio takes the role of the Cauchy-Schwarz inequality, which is not always applicable. ("CS" is an acronym of "Cauchy-Schwarz".)

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With the notion of CS-ratio, the space of equivalence classes of a suitable equivalence relation, termed rays, provides a framework which carries a type of convex geometry. The study of this geometry was initiated in [4], focusing on the so called *quasilinear stars*. The present paper proceeds to develop this theory, employing mostly special characteristic functions, called CS-functions, that emerge from the CS-ratio on ray spaces. These CS-functions provide a useful tool for convex analysis, which is of much help in understanding the variety of quasilinear stars in the ray space.

Supertropical modules are modules over supertropical semirings, which carry a rich algebraic structure [2, 3, 5, 12, 13, 14], and are at the heart of our framework. A supertropical semiring ([6, Definition 0.3]) is a semiring $R$ with idempotent element $e := 1 + 1$ (i.e., $e+e = e$) such that, for all $a, b \in R$, $a+b \in \{a, b\}$ whenever $ea \neq eb$ and $a+b = ea$ otherwise. The ideal $eR$ of $R$ is a bipotent semiring (with unit element $e$), i.e., $a + b$ is either $a$ or $b$, for any $a, b \in eR$. The total ordering

$$a \leqslant b \quad \Leftrightarrow \quad a + b = b$$

of $eR$, together with the ghost map $\nu : a \mapsto ea$, induces the $\nu$-ordering

$$a <_\nu b \quad \Leftrightarrow \quad ea < eb \tag{1.1}$$

and the $\nu$-equivalence

$$a \cong_\nu b \quad \Leftrightarrow \quad ea = eb \tag{1.2}$$

on the entire semiring $R$, which determines the addition of $R$:

$$a + b = \begin{cases} b & \text{if } a <_\nu b, \\ a & \text{if } a >_\nu b, \\ eb & \text{if } a \cong_\nu b. \end{cases}$$

Consequently, $ea = 0 \Rightarrow a = 0$, and the zero $0 = e0$ is regarded mainly as a ghost. The set $\mathcal{T} := R \backslash (eR)$ consists of the **tangible** elements of $R$, while the ideal $\mathcal{G} := (eR) \backslash \{0\}$ contains the **ghost** elements. The semiring $R$ itself is said to be **tangible**, if $e\mathcal{T} = \mathcal{G}$, i.e., $R$ is generated by $\mathcal{T}$ as a semiring. Then, for $\mathcal{T} \neq \emptyset$, $R' := \mathcal{T} \cup e\mathcal{T} \cup \{0\}$ is the largest tangible sub-semiring of $R$.

An $R$-module $V$ over a commutative supertropical semiring $R$ is defined in the familiar way. A **quadratic form** on $V$ is a function $q : V \to R$ satisfying

$$q(ax) = a^2 q(x)$$

for any $a \in R$, $x \in V$, for which there exists a symmetric bilinear form $b : V \times V \to R$, called a **companion** of $q$, such that

$$q(x + y) = q(x) + q(y) + b(x, y)$$

for any $x, y \in V$. ($q$ may have several companions.) The pair $(q, b)$ is called a **quadratic pair**. It is called **balanced**, and $b$ is said to be a **balanced companion** of $q$, if $b(x, x) = eq(x)$ for any $x \in V$.

In our version of "*tropical trigonometry*" the familiar formula $\cos(x,y) = \frac{\langle x,y\rangle}{\|x\|\,\|y\|}$ in euclidian geometry is replaced by the CS-ratio

$$\mathrm{CS}(x,y) := \frac{eb(x,y)^2}{eq(x)q(y)} \in eR \tag{1.3}$$

of **anisotropic** vectors $x, y \in V$, i.e., $q(x) \neq 0$, $q(y) \neq 0$. (As for any supertropical semiring the map $\lambda \mapsto \lambda^2$ is an injective endomorphism, there is no loss of information by squaring $\mathrm{CS}(x,y)$ [6, Proposition 0.5].) The function $x \mapsto \mathrm{CS}(x,w)$ is subadditive for any anisotropic vector $w \in V$ (Theorem 1.1).

In this setting, features of noneuclidian geometry arise, since, not like in euclidian geometry, the CS-ratio $\mathrm{CS}(x,y)$ may take values larger than $e$. These features are closely related to excessiveness [7, Definition 2.8]. When $eR$ is densely ordered, a pair $(x,y)$ is excessive if $\mathrm{CS}(x,y) > e$. When $eR$ is discrete, $(x,y)$ is excessive if either $\mathrm{CS}(x,y) > c_0$, with $c_0$ the smallest element of $eR$ larger than $e$, or $\mathrm{CS}(x,y) = c_0$ and $q(x)$ or $q(y)$ is tangible. A pair $(x,y)$ is **exotic quasilinear**, if $\mathrm{CS}(x,y) = c_0$ and both $q(x)$ and $q(y)$ are ghost [7, Theorems 2.7 and 2.14].

A pair of vectors $(x,y)$ is called **$\nu$-excessive** (resp. **$\nu$-quasilinear**), if the pair $(ex, ey)$ is excessive (resp. quasilinear). Since $\mathrm{CS}(x,y) = \mathrm{CS}(ex,ey)$, it is often simpler to work with $\nu$-excessiveness and $\nu$-quasilinearity. To wit, $(x,y)$ is $\nu$-quasilinear, if

$$q(x,y) \cong_\nu q(x) + q(y),$$

and is $\nu$-excessive otherwise. When $eR$ is dense, $(x,y)$ is $\nu$-quasilinear iff $\mathrm{CS}(x,y) \leqslant e$, while for $eR$ discrete, $\nu$-quasilinear iff $\mathrm{CS}(x,y) \leqslant c_0$; it is exotic quasilinear iff $\mathrm{CS}(x,y) = c_0$.

The CS-ratio obeys important subadditivity rules, involving $\nu$-excessiveness as well as $\nu$-quasilinearity, which are utilized in this paper.

**Theorem 1.1** ([7, Subadditivity Theorem 3.6])**.** *Let $x, y, w \in V$ be anisotropic vectors.*

a) $\mathrm{CS}(x+y,w) \leqslant \mathrm{CS}(x,w) + \mathrm{CS}(y,w)$.

b) *If $(x,y)$ is $\nu$-excessive and* $\mathrm{CS}(x,w) + \mathrm{CS}(y,w) \neq 0$, *then*

$$\mathrm{CS}(x+y,w) < \mathrm{CS}(x,w) + \mathrm{CS}(y,w).$$

c) *If $(x,y)$ is $\nu$-quasilinear and either* $q(x)\,\mathrm{CS}(y,w) = q(y)\,\mathrm{CS}(x,w)$, *or* $\mathrm{CS}(x,w) = \mathrm{CS}(y,w)$, *or* $q(x) \cong_\nu q(y)$, *then*

$$\mathrm{CS}(x+y,w) = \mathrm{CS}(x,w) + \mathrm{CS}(y,w).$$

On an $R$-module $V$ we use the equivalence relation: $x \sim y$ iff $\lambda x = \mu y$ for some $\lambda, \mu \in R\setminus\{0\}$ (where $\lambda, \mu$ need not be invertible as in the usual projective equivalence), whose classes $X$ are called **rays**. It delivers a projective version of the theory on $V\setminus\{0\}$, cf. [7, §6]. When $x$ and $y$ are anisotropic, the CS-ratio $\mathrm{CS}(x,y)$ depends only on the rays $X, Y$ containing $x, y$, and provides a well defined CS-ratio $\mathrm{CS}(X,Y)$ for anisotropic rays $X, Y$, i.e., rays $X, Y$ in $V\setminus q^{-1}(0)$.

Subadditivity of rays occurs on **intervals** $[X,Y]$ with endpoints $X, Y$, cf §2, as a consequence of Theorem 1.1. A comparison of $\mathrm{CS}(Z,W)$ to $\mathrm{CS}(X,W) + \mathrm{CS}(Y,W)$ for anisotropic ray $Z \in [X,Y]$ and arbitrary $W$ is given by [7, Theorem 7.7], and uniqueness of the boundary of $[X,Y]$ by Theorem 2.2. The **ray space** $\mathrm{Ray}(V)$ of $V$ consists of all rays and carries a natural notion of convexity: A subset $A \subset \text{Ray}(V)$ is **convex**, if $[X, Y] \subset A$ for any $X, Y \in A$. Basics structures of rays and convexity in $\text{Ray}(V)$ are reviewed in §2.

By relying on a fine detailed analysis of the monotonicity behavior of the CS-functions

$$\text{CS}(W, -) : \text{Ray}(V) \longrightarrow eR$$

on a fixed interval $[Y_1, Y_2]$ in $\text{Ray}(V)$, given in §3, CS-profiles on interval are defined in §4. This fine analysis enlarges the scope of results in [4] and determines a partition of $\text{Ray}(V)$ into convex subsets (Theorem 5.6) according to the monotonicity behavior of $\text{CS}(W, -)$ on the intervals $[Y_i, Y_j]$, $1 \leqslant i < j \leqslant m$, for a given finite set of rays $\{Y_1, \dots, Y_m\}$ and $W$ running through $\text{Ray}(V)$.

A pair $(X, Y)$ of rays in $V$ is **quasilinear** (with respect to $q$), if the restriction $q|_{Rx+Ry}$ is quasilinear for any $x \in X$, $y \in Y$. A subset $C \subset \text{Ray}(V)$ is **quasilinear**, if all pairs $(X, Y)$ in $C$ are quasilinear. Quasilinearity is governed by **QL-stars** $\text{QL}(X)$ of rays $X$. $\text{QL}(X)$ is the set of all $Y \in \text{Ray}(V)$ for which the pair $(X, Y)$ is quasilinear[1]; equivalently, the interval $[X, Y]$ is quasilinear. §6 presents the **downset** of a QL-star, this is the set of all QL-stars contained in $\text{QL}(X)$, while §7 introduces the **median** on an interval and links it to convexity properties (Corollary 7.6).

The study of median, leads in §8 to enquire the existence of extrema of CS-functions $\text{CS}(W, -)$. Theorem 8.1 specifies a condition and the place where an $eR$-valued function has a minimal, while Theorem 8.4 provides an upper bound in terms of generators for CS-functions over finitely generated $eR$-modules.

Inquiring after the minima of a CS-function is then a natural question. An intriguing issue is that the minimum of $\text{CS}(W, -)$ over the convex hull $\text{conv}(Y_1, \dots, Y_n)$ of $\{Y_1, \dots, Y_n\}$ can be smaller than $\min_i \text{CS}(W, Y_i)$. The rays for which this holds compose the "**glen**" of $Y_1, \dots, Y_n$, which is discussed in detail in §9. Glens extend to intervals, and establish a useful correspondence to CS-functions (Theorem 9.4).

Given a finitely generated convex set $C$ in a ray space $\text{Ray}(V)$, we may ask whether there exists a quadratic pair $(q, b)$ on $V$ with $q$ anisotropic on $V$. In case that $(q, b)$ exists we can move a ray $W$ around, examine the minima of $\text{CS}(W, -)$ on $C$. In the easiest case that $q$ is quasilinear on $C$, the following holds. Every ray $Z$ which is an isolated minimum of a CS-function $\text{CS}(W, -)$, i.e., this function is not constant on an interval emanating from $Z$, is an "indispensable generator" of $C$, i.e., $Z$ occurs in every set of generators of $C$. If $C = \text{conv}(S)$ is the convex hull of some finite set $S$ for which certain pairs $(Z, Z')$ in $S$ are $\nu$-excessive, then the situation is more involved, since $\text{CS}(W, -)$ can be non monotonic on $[Z, Z']$ and the minimum can be attained at the $W$-median. Nevertheless, this gives a constraint, on, say, the minimal sets of generators of $C$. An intriguing phenomenon is that one can choose $W$ nearly arbitrarily.

Motivated by this phenomenon, §10 explores the set $\text{Min}\,\text{CS}(W, C)$ of minima of a CS-function $\text{CS}(W, -)$ on the convex hull $C$ of a finite set $S$ of rays in $\text{Ray}(V)$. Theorem 10.1 characterizes properties of $\text{Min}\,\text{CS}(W, C)$, linking these properties to medians. A $Z$-polar of a subset $P \subset Z^{\uparrow} = \{Y \mid \text{CS}(W, Y) > \text{CS}(W, Z)\}$ is a set of rays (Definition 10.7) for which there exist $X \in P$ with $Y \in Z^{\uparrow}$ such that $M_W(X, Y) = Z$. This subset is closed for taking

[1] $\text{QL}(X)$ is not necessarily quasilienar.

convex hull (Theorem 10.9), compatible with the ordering induced by $Z$ (Theorem 10.10), and induces the $Z$-equivalence relation on $Z^{\uparrow}$. The next step of this study is then to describe the classes of this equivalence relation (Problem 11.1). In this paper we give only a partial description in terms of convex hulls (Theorem 11.2), but, by introducing the notion of **median stars** in §11, we lay out a possible machinery to address this problem.

## 2. Convex sets in the ray space

We review our setup as was laid out in [7, 4] in which $V$ denotes an $R$-module, where $R$ a supertropical semiring $R$ such that $eR$ is a (bipotent) semifield and $R\setminus\{0\}$ is closed for multiplication, i.e., $\lambda\mu = 0 \Rightarrow \lambda = 0$ or $\mu = 0$, for any $\lambda, \mu \in R$. $V$ is assumed to have the property $\lambda x = 0 \Rightarrow \lambda = 0$ or $x = 0$, for any $x \in V$. These properties hold when $eR = \mathcal{G} \cup \{0\}$ is a semifield.

Vectors $x, y \in V$ are **ray-equivalent**, written $x \sim_{\mathrm{r}} y$, if $\lambda x = \mu y$ for some $\lambda, \mu \in R\setminus\{0\}$. This is the finest equivalence relation $E$ on $V$ with $x \sim_E \lambda x$ for any $\lambda \in R\setminus\{0\}$, which gives $V\setminus\{0\}$ as a union of ray-equivalence classes. The **rays** in $V$ are the ray-equivalence classes $\neq \{0\}$. The **ray** $\mathrm{ray}_V(x)$ of $x$ in $V$ is the ray-equivalence class of a vector $x \in V\setminus\{0\}$, written $\mathrm{ray}(x)$ when $V$ is clear form the context. The **ray space** $\mathrm{Ray}(V)$ of $V$ is the set of all rays in $V$. The set $X_0 := X \cup \{0\}$ is the smallest submodule of $V$ containing the ray $X$. $X_0 + Y_0$ is the smallest submodule of $V$ containing both rays $X$ and $Y$. It is a disjoint union of subsemigroups of $(V, +)$ as follows

$$X_0 + Y_0 = (X + Y) \cup X \cup Y \cup \{0\}. \tag{2.1}$$

The **closed interval** $[X, Y]$ consists of all rays $Z$ in the submodule $X_0 + Y_0$ of $V$, generated by $X \cup Y$. The **open interval** $]X, Y[$ consists of all rays

$$Z \subset X + Y := \{x + y \mid x \in X, y \in Y\}.$$

Thus, $[X, Y] = \ ]X, Y[ \ \cup \{X, Y\}$. The **half open intervals** are

$$[X, Y[ \ := \ ]X, Y[ \ \cup \{X\}, \qquad ]X, Y] := \ ]X, Y[ \ \cup \{Y\}.$$

**Scholium 2.1** ([7, Scholium 7.6])**.** *Set $X = \mathrm{ray}(x)$, $Y = \mathrm{ray}(y)$ for $x, y \in V$. For any ray $Z$ in $V$, the following hold:*

a) *$Z \in \ ]X, Y[$ iff $Z = \mathrm{ray}(\lambda x + \mu y)$ with $\lambda, \mu \in R\setminus\{0\}$ iff $Z = \mathrm{ray}(\lambda x + y)$ with $\lambda \in R\setminus\{0\}$.*

b) *$Z \in \ ]X, Y]$ iff $Z = \mathrm{ray}(\lambda x + \mu y)$ with $\lambda \in R$, $\mu \in R\setminus\{0\}$ iff $Z = \mathrm{ray}(\lambda x + y)$ with $\lambda \in R$.*

c) *$Z \in [X, Y]$ iff $Z = \mathrm{ray}(\lambda x + \mu y)$ with $\lambda, \mu \in R$.*

*$R$ and $R\setminus\{0\}$ may be replaced respectively by $eR$ and $\mathcal{G}$ everywhere.*

**Theorem 2.2** ([7, Theorem 8.8])**.** *Let $X, Y, X_1, Y_1$ be rays in $V$ with $[X, Y] = [X_1, Y_1]$.*

a) *Either $X = X_1$, $Y = Y_1$ or $X = Y_1$, $Y = X_1$.*

b) *If $[X, Y]$ is not a singleton, i.e., $X \neq Y$, then $X = X_1$, $Y = Y_1$ iff $[X, X_1] \neq [X, Y_1]$.*

A subset $M \subset \mathrm{Ray}(V)$ is **convex** (in $\mathrm{Ray}(V)$), if for any two rays $X, Y \in M$ the closed interval $[X, Y]$ is contained in $M$. The **convex hull** $\mathrm{conv}(S)$ of a nonempty set $S \subset \mathrm{Ray}(V)$ is the smallest convex subset of $\mathrm{Ray}(V)$ containing $S$. When $S = \{X_1, \dots, X_n\}$ is finite,

$\operatorname{conv}(S)$ is written $\operatorname{conv}(X_1, \dots, X_n)$, for short. Clearly $[X, Y] = \operatorname{conv}(X, Y)$, and by [7, Proposition 8.1] all the intervals $]X, Y[\,, ]X, Y], [X, Y[\,, [X, Y]$ are convex sets, for any rays $X, Y$ in $\operatorname{Ray}(V)$. A subset $U \subset V$ is ***ray-closed*** in $V$, if $U \backslash \{0\}$ is a union of rays of $V$.

**Proposition 2.3** ([4, Proposition 2.6]).

(a) *If $U_1, \dots, U_n$ are ray-closed subsets of $V \backslash \{0\}$, then the set $U_1 + \cdots + U_n$ is again ray-closed in $V$, consisting of all rays $\operatorname{ray}_V(\lambda_1 u_1 + \cdots + \lambda_n u_n)$ with $u_i \in U_i$, $\lambda_i \in R\backslash\{0\}$. In particular, for any rays $X_1, \dots, X_n$ in $V$ the set $X_1 + \cdots + X_n$ is ray-closed in $V$.*

(b) *The convex hull of a finite set of rays $\{X_1, \dots, X_n\}$ has the disjoint decomposition*
$$\operatorname{conv}(X_1, \dots, X_n) = \bigcup_{i_1 < \cdots < i_r} \operatorname{Ray}(X_{i_1} + \cdots + X_{i_r})$$
*with $r \leqslant n$, $1 \leqslant i_1 < \cdots < i_r \leqslant n$.*

We denote by $\widetilde{A}$ the set of all sums of finitely many members of a subset $A \subset \operatorname{Ray}(V)$. As the convex hull of $A$ is the union of all sets $\operatorname{conv}(X_1, \dots, X_r)$ with $r \in \mathbb{N}$, $X_1, \dots, X_r \in A$, we obtain the following.

**Corollary 2.4** ([4, Corollary 2.8]). *Let $C$ denote the convex hull of $A_1 \cup \cdots \cup A_n$, where $A_1, \dots, A_n$ are convex subsets of $\operatorname{Ray}(V)$.*

(a) *$C$ is the union of all convex hulls $\operatorname{conv}(X_1, \dots, X_n)$ with $X_i \in A_i$, $1 \leqslant i \leqslant n$.*

(b) *$\widetilde{C}$ is the union of all sets $\widetilde{A}_{i_1} + \cdots + \widetilde{A}_{i_r}$ with $r \leqslant n$, $1 \leqslant i_1 < \cdots < i_r \leqslant n$.*

Proposition 2.3 and Corollary 2.4 can be inferred from the next observation.

**Proposition 2.5** ([4, Proposition 2.9]). *The convex subsets $A$ of $\operatorname{Ray}(V)$ correspond uniquely to the ray-closed submodules $W$ of $V$ via $W = \widetilde{A} \cup \{0\}$, $A = \operatorname{Ray}(W)$.*

## 3. The function $\operatorname{CS}(X_1, -)$ on $[X_2, X_3]$

In this section $R$ denotes a supertropical semiring whose ghost ideal $eR = \{0\} \cup \mathcal{G}$ is a nontrivial (bipotent) semifield, and $V$ stands for an $R$-module equipped with a fixed quadratic pair $(q, b)$ with $q$ anisotropic on $V$, i.e., $q^{-1}(0) = \{0\}$. Assuming that $X_1, X_2, X_3$ are three rays on $V$, we explicitly analyze the monotonicity behavior of the function $\operatorname{CS}(X_1, -)$ on the interval $[X_2, X_3]$ with $X_2 \neq X_3$. For vectors $\varepsilon_i \in X_i$, $i = 1, 2, 3$, we employ the following six parameters
$$\alpha_i = q(\varepsilon_i) \neq 0, \quad \alpha_{ij} = b(\varepsilon_i, \varepsilon_j) = \alpha_{ji} \qquad i, j \in \{1, 2, 3\},\ i < j.$$
As our computations take place in the semifield $eR$, we can write $\leqslant, <$ instead of $\leqslant_\nu, <_\nu$. But, the forthcoming formulas are to be used later in a supertropical context without fuss, otherwise we could assume that the parameters $\alpha_i, \alpha_{ij}$ belong to $eR$.

Our analysis is performed in terms of the function
$$f(\lambda) = \operatorname{CS}(\varepsilon_1, \varepsilon_2 + \lambda \varepsilon_3)$$
with $\lambda \in eR \cup \{\infty\} = \{0\} \cup \mathcal{G} \cup \{\infty\}$. Here $\lambda = \infty$ corresponds to $\mu = 0$ for $\mu = \lambda^{-1}$, and $\operatorname{CS}(\varepsilon_1, \varepsilon_2 + \lambda\varepsilon_3) = \operatorname{CS}(\varepsilon_1, \mu\varepsilon_2 + \varepsilon_3)$, thus $f(\infty) = \operatorname{CS}(\varepsilon_1, \varepsilon_3)$. We have
$$b(\varepsilon_1, \varepsilon_2 + \lambda\varepsilon_3) = \alpha_{12} + \lambda\alpha_{13},$$

and so

$$f(\lambda) = e\frac{\alpha_{12}^2 + \lambda^2\alpha_{13}^2}{\alpha_1\, q(\varepsilon_2 + \lambda\varepsilon_3)} \in eR, \tag{3.1}$$

which decomposes as

$$f(\lambda) = f_1(\lambda) + f_2(\lambda) = \max(f_1(\lambda), f_2(\lambda)) \tag{3.2}$$

with

$$f_1(\lambda) = e\frac{\alpha_{12}^2}{\alpha_1\, q(\varepsilon_2 + \lambda\varepsilon_3)}, \qquad f_2(\lambda) = e\frac{\lambda^2\alpha_{13}^2}{\alpha_1\, q(\varepsilon_2 + \lambda\varepsilon_3)}.^2 \tag{3.3}$$

We proceed by analysing the monotonicity behavior of $f_1, f_2$ on $[0,\infty]$. Without loss of generality we assume that

$$\mathrm{CS}(\varepsilon_1, \varepsilon_2) \leqslant \mathrm{CS}(\varepsilon_1, \varepsilon_3). \tag{3.4}$$

(Otherwise interchange $X_2$ and $X_3$.)

If $\mathrm{CS}(\varepsilon_1, \varepsilon_3) = 0$, then $\alpha_{12} = \alpha_{13} = 0$, and thus $f_1 = 0$, $f_2 = 0$, $f = 0$. *Discarding this trivial case, we assume that* $\mathrm{CS}(\varepsilon_1, \varepsilon_3) > 0$. We rewrite the functions $f_1, f_2$ as follows

$$f_1(\lambda) = \frac{\alpha_{12}^2}{\alpha_1\alpha_2}\,\frac{\alpha_2}{q(\varepsilon_2 + \lambda\varepsilon_3)} = \mathrm{CS}(\varepsilon_1, \varepsilon_2)\frac{q(\varepsilon_2)}{q(\varepsilon_2 + \lambda\varepsilon_3)}, \tag{3.5}$$

$$f_2(\lambda) = \frac{\alpha_{13}^2}{\alpha_1\alpha_3}\,\frac{\lambda^2\alpha_3}{q(\varepsilon_2 + \lambda\varepsilon_3)} = \mathrm{CS}(\varepsilon_1, \varepsilon_3)\frac{q(\lambda\varepsilon_3)}{q(\varepsilon_2 + \lambda\varepsilon_3)}. \tag{3.6}$$

These formulas imply that

$$f_1(\lambda) \leqslant \mathrm{CS}(\varepsilon_1, \varepsilon_2), \quad f_2(\lambda) \leqslant \mathrm{CS}(\varepsilon_1, \varepsilon_3), \qquad \text{for all } \lambda \in [0,\infty]. \tag{3.7}$$

More explicitly,

$$q(\varepsilon_2 + \lambda\varepsilon_3) = \alpha_2 + \lambda\alpha_{23} + \lambda^2\alpha_3, \tag{3.8}$$

and so

$$f_1(\lambda) = \mathrm{CS}(\varepsilon_1, \varepsilon_2)\frac{\alpha_2}{\alpha_2 + \lambda\alpha_{23} + \lambda^2\alpha_3}, \tag{3.9}$$

$$f_2(\lambda) = \mathrm{CS}(\varepsilon_1, \varepsilon_3)\frac{\alpha_3}{\alpha_3 + \lambda^{-1}\alpha_{23} + \lambda^{-2}\alpha_2}. \tag{3.10}$$

We conclude from these formulas that $f_1$ decreases (monotonically) on $[0,\infty]$ from $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$ to zero, while $f_2$ increases (monotonically) from zero to $\mathrm{CS}(\varepsilon_1, \varepsilon_3)$. Moreover, we infer from (3.3) that $f_1(\lambda) = f_2(\lambda)$ precisely when $\alpha_{12}^2 = \lambda^2\alpha_{13}^2$, whence $\lambda^2 = \frac{\alpha_{12}^2}{\alpha_{13}^2}$. So $f_1(\lambda) = f_2(\lambda)$ holds on the unique argument $\xi$, that is

$$\xi = \frac{\alpha_{12}}{\alpha_{13}}. \tag{3.11}$$

It follows that $f$ coincides with $f_1$ on $[0,\xi]$ and with $f_2$ on $[\xi,\infty]$. Furthermore $f(\xi) = f_1(\xi) = f_2(\xi)$ is the minimal value attained by the function $f$ on $[0,\infty]$. In other words, $f(\xi)$ is the minimal value of $\mathrm{CS}(X_1, Z)$ for $Z$ running over $[X_2, X_3]$. $\xi$ corresponds to the ray

$$M := \mathrm{ray}\left(\varepsilon_2 + \frac{\alpha_{12}}{\alpha_{13}}\varepsilon_3\right) = \mathrm{ray}(\alpha_{13}\varepsilon_2 + \alpha_{12}\varepsilon_3), \tag{3.12}$$

which we call the $X_1$-**median** of the interval $[X_2, X_3]$. This important ray $M$ will be studied in detail later.

[2] In the following we omit the factor $e = 1_{eR}$, reading all formulas in $eR$.

So far we have obtained an outline of the monotonicity behavior of $f_1, f_2, f$. This picture will now be refined. We start with the case that

$$\alpha_2\alpha_3 \leqslant \alpha_{23}^2, \tag{3.13}$$

which, except in the border case $\alpha_2\alpha_3 = \alpha_{23}^2$, implies that the interval $[X_2, X_3]$ is excessive or exotic quasilinear [7, Definition 2.8]. In particular $\alpha_{23} \neq 0$.

We determine the subsets of $[0,\infty]$ where the decreasing function $f_1$ takes its maximal value $\mathrm{CS}(\varepsilon_1,\varepsilon_2)$ and the increasing function $f_2$ takes its maximal value $\mathrm{CS}(\varepsilon_1,\varepsilon_3)$ as follows. When $\mathrm{CS}(\varepsilon_1,\varepsilon_2) \neq 0$, we read off from Formula (3.7), applied to $q(\varepsilon_2+\lambda\varepsilon_3)$, that $f_1(\lambda) = \mathrm{CS}(\varepsilon_1,\varepsilon_2)$ precisely when the summand $\alpha_2$ is $\nu$-dominant, i.e., $\alpha_2 \geqslant \lambda\alpha_{23}$, $\alpha_2 \geqslant \lambda^2\alpha_3$, equivalently,

$$\lambda^2 \leqslant \frac{\alpha_2^2}{\alpha_{23}^2}, \qquad \lambda^2 \leqslant \frac{\alpha_2}{\alpha_3}.$$

From (3.13) we infer that $\frac{\alpha_2}{\alpha_3} \geqslant \frac{\alpha_2^2}{\alpha_{23}^2}$, and therefore the condition $\lambda^2 \leqslant \frac{\alpha_2}{\alpha_3}$ can be dismissed. Thus

$$f_1(\lambda) = \mathrm{CS}(\varepsilon_1,\varepsilon_2) \quad \Leftrightarrow \quad \lambda \leqslant \frac{\alpha_2}{\alpha_{23}}. \tag{3.14}$$

The case of $\mathrm{CS}(\varepsilon_1,\varepsilon_2) = 0$ is degenerate, in which $f_1 = 0$, $f_2 = f$ (and $\xi = 0$).

Concerning $f_2$, we read off from (3.6) and (3.10) that $f_2(\lambda) = \mathrm{CS}(\varepsilon_1,\varepsilon_3)$ iff $\lambda \neq 0$ and the term $\alpha_3$ in the sum $\alpha_3 + \lambda^{-1}\alpha_{23} + \lambda^{-2}\alpha_2$ is $\nu$-dominant, which means that $\lambda^{-1}\alpha_{23} \leqslant \alpha_3$, $\lambda^{-2}\alpha_2 \leqslant \alpha_3$, equivalently,

$$\frac{\alpha_{23}^2}{\alpha_3^2} \leqslant \lambda^2, \quad \frac{\alpha_2}{\alpha_3} \leqslant \lambda^2.$$

We conclude from (3.13) that $\frac{\alpha_{23}^2}{\alpha_3^2} \geqslant \frac{\alpha_2}{\alpha_3}$, and thus

$$f_2(\lambda) = \mathrm{CS}(\varepsilon_1,\varepsilon_3) \quad \Leftrightarrow \quad \lambda \geqslant \frac{\alpha_{23}}{\alpha_3}. \tag{3.15}$$

{Recall that we initially assume that $\mathrm{CS}(\varepsilon_1,\varepsilon_3) \neq 0$.}

We have seen that the intervals $[0, \frac{\alpha_2}{\alpha_{23}}]$ and $[\frac{\alpha_{23}}{\alpha_3}, \infty]$ are the sets where the terms $\alpha_1$ and $\lambda^2\alpha_2$ in the sum (3.8) are $\nu$-dominant and conclude that $[\frac{\alpha_2}{\alpha_{23}}, \frac{\alpha_{23}}{\alpha_3}]$ is the interval in which the middle term $\lambda\alpha_{23}$ is $\nu$-dominant. Note that in the border case $\alpha_2\alpha_3 = \alpha_{23}^2$ this interval retracts to the single point $\frac{\alpha_2}{\alpha_{23}} = \frac{\alpha_{23}}{\alpha_3}$.

We infer from (3.6) and (3.10) that in the interval $[0, \frac{\alpha_2}{\alpha_{23}}]$

$$f_2(\lambda) = \mathrm{CS}(\varepsilon_1,\varepsilon_3)\frac{\lambda^2\alpha_3}{\alpha_2}, \tag{3.16}$$

and that in $[\frac{\alpha_2}{\alpha_{23}}, \frac{\alpha_{23}}{\alpha_3}]$

$$f_2(\lambda) = \mathrm{CS}(\varepsilon_1,\varepsilon_3)\frac{\lambda\alpha_3}{\alpha_{23}}. \tag{3.17}$$

Thus $f_2$ strictly increases on $[0, \frac{\alpha_2}{\alpha_{23}}]$ from zero to

$$f_2\left(\frac{\alpha_2}{\alpha_{23}}\right) = \mathrm{CS}(\varepsilon_1,\varepsilon_3)\frac{\alpha_3}{\alpha_2}\left(\frac{\alpha_2}{\alpha_{23}}\right)^2 = \frac{\mathrm{CS}(\varepsilon_1,\varepsilon_3)}{\mathrm{CS}(\varepsilon_2,\varepsilon_3)},$$

and then strictly increases on $[\frac{\alpha_2}{\alpha_{23}}, \frac{\alpha_{23}}{\alpha_3}]$ from this value to $\mathrm{CS}(\varepsilon_1, \varepsilon_3)$. Furthermore, we infer from (3.5) and (3.9) that in the interval $[\frac{\alpha_2}{\alpha_{23}}, \frac{\alpha_{23}}{\alpha_3}]$

$$f_1(\lambda) = \mathrm{CS}(\varepsilon_1, \varepsilon_2)\frac{\alpha_2}{\lambda\alpha_{23}}, \tag{3.18}$$

while in $[\frac{\alpha_{23}}{\alpha_3}, \infty]$

$$f_1(\lambda) = \mathrm{CS}(\varepsilon_1, \varepsilon_2)\frac{\alpha_2}{\lambda^2\alpha_3}. \tag{3.19}$$

Thus $f_1$ strictly decreases on $[\frac{\alpha_2}{\alpha_{23}}, \frac{\alpha_{23}}{\alpha_3}]$ from the value $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$ to

$$f_1\left(\frac{\alpha_{23}}{\alpha_3}\right) = \mathrm{CS}(\varepsilon_1, \varepsilon_2)\frac{\alpha_2}{\alpha_3}\left(\frac{\alpha_3}{\alpha_{23}}\right)^2 = \frac{\mathrm{CS}(\varepsilon_1, \varepsilon_2)}{\mathrm{CS}(\varepsilon_2, \varepsilon_3)}$$

and then on $[\frac{\alpha_{23}}{\alpha_3}, \infty]$ again strictly decreases from this value to zero. Note that the arguments $\lambda = \frac{\alpha_2}{\alpha_{23}}, \frac{\alpha_{23}}{\alpha_3}$ correspond to the rays

$$X_{23} := \mathrm{ray}(\alpha_{23}\varepsilon_2 + \alpha_2\varepsilon_3), \qquad X_{32} := \mathrm{ray}(\alpha_3\varepsilon_2 + \alpha_{23}\varepsilon_3), \tag{3.20}$$

which in the case $\alpha_2\alpha_3 <_\nu \alpha_{23}^2$ are the **critical rays** of $[X_2, X_3]$ (cf. [7]).

Summarizing the above study we obtain:

**Proposition 3.1.** *Assume that* $0 \leqslant \mathrm{CS}(\varepsilon_1, \varepsilon_2) \leqslant \mathrm{CS}(\varepsilon_1, \varepsilon_3)$ *and that* $\alpha_2\alpha_3 \leqslant_\nu \alpha_{23}^2$.

a) *The function* $f_1$ *is constant on* $[0, \frac{\alpha_2}{\alpha_{23}}]$ *with value* $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$ *and strictly decreases on* $[\frac{\alpha_2}{\alpha_{23}}, \infty]$ *from* $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$ *to zero, with the intermediate value*

$$f_1\left(\frac{\alpha_2}{\alpha_{23}}\right) = \frac{\mathrm{CS}(\varepsilon_1, \varepsilon_2)}{\mathrm{CS}(\varepsilon_2, \varepsilon_3)},$$

*provided that* $\mathrm{CS}(\varepsilon_1, \varepsilon_2) \neq 0$. *Otherwise* $f_1 = 0$, *whence* $f_2 = f$ *on* $[0, \infty]$.

b) *The function* $f_2$ *strictly increases on* $[0, \frac{\alpha_{23}}{\alpha_5}]$ *from zero to* $\mathrm{CS}(\varepsilon_1, \varepsilon_3)$ *and remains constant on* $[\frac{\alpha_{23}}{\alpha_3}, \infty]$, *with intermediate value*

$$f_2\left(\frac{\alpha_2}{\alpha_{23}}\right) = \frac{\mathrm{CS}(\varepsilon_1, \varepsilon_3)}{\mathrm{CS}(\varepsilon_2, \varepsilon_3)},$$

*provided that* $\mathrm{CS}(\varepsilon_1, \varepsilon_3) \neq 0$. *Otherwise* $f_1 = f_2 = f = 0$ *on* $[0, \infty]$.

Since $\xi$ is the unique argument $\lambda \in [0, \infty]$ with $f_1(\lambda) = f_2(\lambda)$, it follows from Proposition 3.1 that $f_1 \geqslant f_2$ on $[0, \xi]$ and $f_1 \leqslant f_2$ on $[\xi, \infty]$, whence

$$f = f_1 \text{ on } [0, \xi] \quad \text{and} \quad f = f_2 \text{ on } [\xi, \infty]. \tag{3.21}$$

As seen below, the monotonicity behavior of $f$ is determined by the location of $\xi$ with respect to the interval $[\frac{\alpha_2}{\alpha_{23}}, \frac{\alpha_{23}}{\alpha_3}]$.

Since $f_2 = \mathrm{CS}(\varepsilon_1, \varepsilon_3) \geqslant f_1$ on $[\frac{\alpha_{23}}{\alpha_3}, \infty]$ (cf. (3.7)), it is clear that always

$$\xi \leqslant \frac{\alpha_{23}}{\alpha_3}. \tag{3.22}$$

We have $\xi = 0$ iff $\mathrm{CS}(\varepsilon_1, \varepsilon_2) = 0$, and then $f = f_2$ strictly increases on $[0, \frac{\alpha_{23}}{\alpha_3}]$ to $\mathrm{CS}(\varepsilon_1, \varepsilon_3)$ and remains constant on $[\frac{\alpha_{23}}{\alpha_3}, \infty]$.

Assuming that $\mathrm{CS}(\varepsilon_1, \varepsilon_2) > 0$, if $\xi \leqslant \frac{\alpha_2}{\alpha_{23}}$, then $f$ has the constant value $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$ on $[0, \xi]$. Thus, as follows from Proposition 3.1, $f$ strictly increases on $[\xi, \frac{\alpha_2}{\alpha_{23}}]$ from $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$

to $\frac{\mathrm{CS}(\varepsilon_1,\varepsilon_3)}{\mathrm{CS}(\varepsilon_1,\varepsilon_2)}$, and it strictly increases on $[\frac{\alpha_2}{\alpha_{23}}, \frac{\alpha_{23}}{\alpha_3}]$ from this value to $\mathrm{CS}(\varepsilon_1, \varepsilon_3)$. Finally, $f$ remains constant on $[\frac{\alpha_{23}}{\alpha_3}, \infty]$.

The graph of the function $f$ is illustrated as follows.

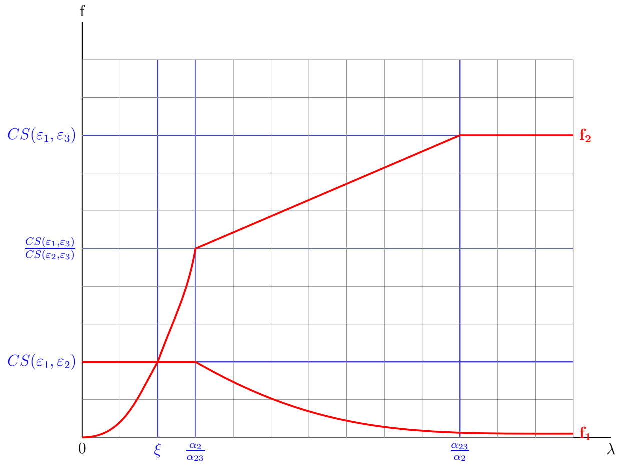


FIGURE 1.

We read off from this analysis that

$$\xi < \frac{\alpha_2}{\alpha_{23}} \quad \Leftrightarrow \quad \mathrm{CS}(\varepsilon_1, \varepsilon_2) < \frac{\mathrm{CS}(\varepsilon_1, \varepsilon_2)}{\mathrm{CS}(\varepsilon_2, \varepsilon_3)}, \tag{3.23}$$

$$\xi = \frac{\alpha_2}{\alpha_{23}} \quad \Leftrightarrow \quad \mathrm{CS}(\varepsilon_1, \varepsilon_2) = \frac{\mathrm{CS}(\varepsilon_1, \varepsilon_3)}{\mathrm{CS}(\varepsilon_2, \varepsilon_3)}. \tag{3.24}$$

In the remaining case that

$$\frac{\alpha_2}{\alpha_{23}} < \xi \leqslant \frac{\alpha_{23}}{\alpha_2}$$

we conclude by Proposition 3.1 and (3.21) that $f$ has the constant value $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$ on $[0, \frac{\alpha_2}{\alpha_{23}}]$, then strictly decreases on $[\frac{\alpha_2}{\alpha_{23}}, \xi]$ to a value $\rho := f_1(\xi) = f_2(\xi)$ which we compute below. Then $f$ strictly increases on $[\xi, \frac{\alpha_{23}}{\alpha_3}]$ from the value $\rho$ to $\mathrm{CS}(\varepsilon_1, \varepsilon_3)$, and finally remains constant on $[\frac{\alpha_{23}}{\alpha_3}, \infty]$. This implies that

$$\xi < \frac{\alpha_{23}}{\alpha_3}, \tag{3.22'}$$

improving (3.22). By (3.11) and (3.17) we have $\rho = f_2(\xi) = \mathrm{CS}(\varepsilon_1, \varepsilon_3)\frac{\alpha_{12}}{\alpha_{13}}\frac{\alpha_2}{\alpha_{23}}$, yielding

$$\rho = \frac{\alpha_{12}\alpha_{13}}{\alpha_1\alpha_{23}}, \tag{3.25}$$

whose square gives

$$\rho^2 = \frac{\alpha_{12}^2\alpha_{13}^2}{\alpha_1^2\alpha_{23}^2} = \frac{\alpha_{12}^2}{\alpha_1\alpha_2}\frac{\alpha_{13}^2}{\alpha_1\alpha_3}\frac{\alpha_2\alpha_3}{\alpha_{23}^2},$$

i.e.,

$$\rho^2 = \frac{\mathrm{CS}(\varepsilon_1, \varepsilon_2)\,\mathrm{CS}(\varepsilon_1, \varepsilon_3)}{\mathrm{CS}(\varepsilon_2, \varepsilon_3)}. \tag{3.26}$$

It is now clear that $\xi > \frac{\alpha_2}{\alpha_{23}}$ iff $f_2(\xi) < \mathrm{CS}(\varepsilon_1, \varepsilon_2)$ iff

$$\frac{\mathrm{CS}(\varepsilon_1, \varepsilon_2)\,\mathrm{CS}(\varepsilon_1, \varepsilon_3)}{\mathrm{CS}(\varepsilon_2, \varepsilon_3)} < \mathrm{CS}(\varepsilon_1, \varepsilon_2)^2,$$

and thus

$$\xi > \frac{\alpha_2}{\alpha_{23}} \quad \Leftrightarrow \quad \frac{\mathrm{CS}(\varepsilon_1, \varepsilon_3)}{\mathrm{CS}(\varepsilon_2, \varepsilon_3)} < \mathrm{CS}(\varepsilon_1, \varepsilon_2). \tag{3.27}$$

Recall that we have assumed (cf. (3.4)) that $0 < \mathrm{CS}(\varepsilon_1, \varepsilon_2) \leqslant \mathrm{CS}(\varepsilon_1, \varepsilon_3)$. Assuming further that

$$\frac{\mathrm{CS}(\varepsilon_1, \varepsilon_3)}{\mathrm{CS}(\varepsilon_2, \varepsilon_3)} < \mathrm{CS}(\varepsilon_1, \varepsilon_2), \tag{3.28}$$

we still need to distinguish the cases $\mathrm{CS}(\varepsilon_1, \varepsilon_2) < \mathrm{CS}(\varepsilon_1, \varepsilon_3)$ and $\mathrm{CS}(\varepsilon_1, \varepsilon_2) = \mathrm{CS}(\varepsilon_1, \varepsilon_3)$ (where (3.28) holds automatically).

This means that either $f(0) < f(\infty)$ or $f(0) = f(\infty)$, which we judge as a difference in the monotonicity behavior of $f$. The graph of $f$ is illustrated in Figure 2.

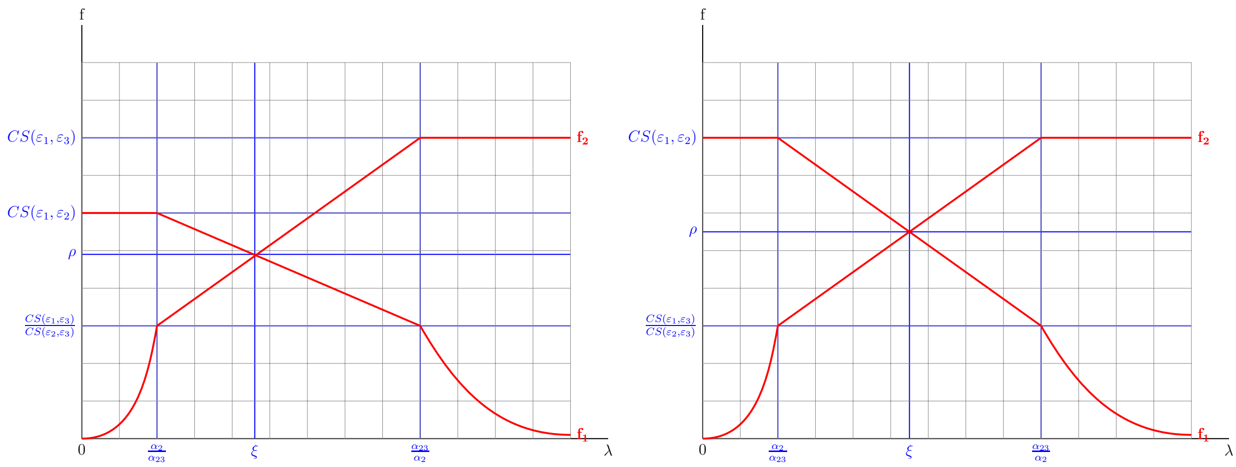


FIGURE 2. A. $(\mathrm{CS}(\varepsilon_1, \varepsilon_2) < \mathrm{CS}(\varepsilon_1, \varepsilon_3))$, B. $(\mathrm{CS}(\varepsilon_1, \varepsilon_2) = \mathrm{CS}(\varepsilon_1, \varepsilon_3))$.

We summarize the above analysis of $f(\lambda) = \mathrm{CS}(\varepsilon_1, \varepsilon_2 + \lambda\varepsilon_2)$ for $\mathrm{CS}(\varepsilon_2, \varepsilon_3) > e$ as follows.

**Theorem 3.2.** *Assume that $0 \leqslant \mathrm{CS}(\varepsilon_1, \varepsilon_2) \leqslant \mathrm{CS}(\varepsilon_1, \varepsilon_3)$ and that $\alpha_2\alpha_3 <_\nu \alpha_{23}^2$.*

a) *If $\mathrm{CS}(\varepsilon_1, \varepsilon_3) = 0$, then $f = 0$. If $\mathrm{CS}(\varepsilon_1, \varepsilon_2) = 0$ and $\mathrm{CS}(\varepsilon_1, \varepsilon_3) > 0$, then $f = 0$ on $[0, \frac{\alpha_2}{\alpha_{23}}]$, but $f$ strictly increases on $[\frac{\alpha_2}{\alpha_{23}}, \frac{\alpha_{23}}{\alpha_3}]$ from zero to $\mathrm{CS}(\varepsilon_1, \varepsilon_3)$ and finally remains constant on $[\frac{\alpha_{23}}{\alpha_3}, \infty]$.*

b) *Assume now that*

$$0 < \mathrm{CS}(\varepsilon_1, \varepsilon_2) \leqslant \mathrm{CS}(\varepsilon_1, \varepsilon_3).$$

*In the case*

$$\mathrm{CS}(\varepsilon_1, \varepsilon_2) < \frac{\mathrm{CS}(\varepsilon_1, \varepsilon_3)}{\mathrm{CS}(\varepsilon_2, \varepsilon_3)} < \mathrm{CS}(\varepsilon_1, \varepsilon_3) \tag{A}$$

*the function $f$ increases on $[0, \infty]$ monotonically from $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$ to $\mathrm{CS}(\varepsilon_1, \varepsilon_3)$. More precisely, $\frac{\alpha_{12}}{\alpha_{13}} < \frac{\alpha_2}{\alpha_{23}}$, and $f$ has constant value $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$ on $[0, \frac{\alpha_{12}}{\alpha_{13}}]$, then it strictly* *increases on* $[\xi, \frac{\alpha_2}{\alpha_{23}}]$ *to the value* $\frac{\mathrm{CS}(\varepsilon_1,\varepsilon_3)}{\mathrm{CS}(\varepsilon_2,\varepsilon_3)}$ *and strictly increases again on* $[\frac{\alpha_2}{\alpha_{23}}, \frac{\alpha_{23}}{\alpha_3}]$ *to the value* $\mathrm{CS}(\varepsilon_1, \varepsilon_3)$ *for which it remains constant on* $[\frac{\alpha_{23}}{\alpha_3}, \infty]$.

*In the border case*

$$\mathrm{CS}(\varepsilon_1, \varepsilon_2) = \frac{\mathrm{CS}(\varepsilon_1, \varepsilon_3)}{\mathrm{CS}(\varepsilon_2, \varepsilon_3)}, \tag{$\partial A$}$$

*we have* $\frac{\alpha_{12}}{\alpha_{13}} = \frac{\alpha_2}{\alpha_{23}}$, *where* $f$ *is constant of value* $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$ *on* $[0, \frac{\alpha_2}{\alpha_{23}}]$, *strictly increases on* $[\frac{\alpha_2}{\alpha_{23}}, \frac{\alpha_{23}}{\alpha_3}]$ *from* $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$ *to* $\mathrm{CS}(\varepsilon_1, \varepsilon_3)$, *and finally is constant of value* $\mathrm{CS}(\varepsilon_1, \varepsilon_3)$ *on* $[\frac{\alpha_{23}}{\alpha_3}, \infty]$.

*In the remaining case*

$$\frac{\mathrm{CS}(\varepsilon_1, \varepsilon_3)}{\mathrm{CS}(\varepsilon_2, \varepsilon_3)} < \mathrm{CS}(\varepsilon_1, \varepsilon_2) < \mathrm{CS}(\varepsilon_1, \varepsilon_3) \tag{B}$$

*and its border case*

$$0 < \mathrm{CS}(\varepsilon_1, \varepsilon_2) = \mathrm{CS}(\varepsilon_1, \varepsilon_3) \tag{$\partial B$}$$

*the function* $f$ *attains its minimal value* $\rho$ *at the unique point* $\lambda = \xi = \frac{\alpha_{12}}{\alpha_{13}}$, *where*

$$\frac{\alpha_2}{\alpha_{23}} < \xi < \frac{\alpha_{23}}{\alpha_3}.$$

*Explicitly,* $f$ *has the constant value* $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$ *on* $[0, \frac{\alpha_2}{\alpha_{23}}]$, *it strictly decreases on* $[\frac{\alpha_2}{\alpha_{23}}, \xi]$ *to the value*

$$\rho := \frac{\alpha_{12}\alpha_{13}}{\alpha_1\alpha_{23}} = \sqrt{\frac{\mathrm{CS}(\varepsilon_1, \varepsilon_2)\,\mathrm{CS}(\varepsilon_1, \varepsilon_3)}{\mathrm{CS}(\varepsilon_2, \varepsilon_3)}},$$

*then it strictly increases on* $[\xi, \frac{\alpha_{23}}{\alpha_3}]$ *to the value* $\mathrm{CS}(\varepsilon_1, \varepsilon_3)$ *and remains constant on* $[\frac{\alpha_{23}}{\alpha_3}, \infty]$.

Finally we discuss the behavior of $f(\lambda)$ in the easier case that $\alpha_{23}^2 \leqslant_\nu \alpha_2\alpha_3$, assuming as before that $0 \leqslant \mathrm{CS}(\varepsilon_1, \varepsilon_2) \leqslant \mathrm{CS}(\varepsilon_1, \varepsilon_3)$. Formula (3.8) for $q(\varepsilon_2 + \lambda\varepsilon_3)$ simplifies to

$$q(\varepsilon_2 + \lambda\varepsilon_3) = \alpha_2 + \lambda^2\alpha_3,$$

and so

$$f_1(\lambda) = \frac{\alpha_{12}^2}{\alpha_1(\alpha_2 + \lambda^2\alpha_3)}, \qquad f_2(\lambda) = \frac{\lambda^2\alpha_{13}^2}{\alpha_1(\alpha_2 + \lambda^2\alpha_3)}. \tag{3.29}$$

If $\mathrm{CS}(\varepsilon_1, \varepsilon_3) = 0$, i.e., $\alpha_{13} = 0$, then $f_1 = 0$, $f_2 = 0$, $f = 0$. Henceforth we assume that $\mathrm{CS}(\varepsilon_1, \varepsilon_3) > 0$ (but allow that $\mathrm{CS}(\varepsilon_1, \varepsilon_2) = 0$). We read off from (3.29) that, if $\lambda^2 \leqslant \frac{\alpha_2}{\alpha_3}$, then

$$f_1(\lambda) = \frac{\alpha_{12}^2}{\alpha_1\alpha_2} = \mathrm{CS}(\varepsilon_1, \varepsilon_2), \qquad f_2(\lambda) = \frac{\lambda^2\alpha_{13}^2}{\alpha_1\alpha_2} = \lambda^2\frac{\alpha_3}{\alpha_2}\,\mathrm{CS}(\varepsilon_1, \varepsilon_3), \tag{3.30}$$

while, if $\lambda^2 \geqslant \frac{\alpha_2}{\alpha_3}$, then

$$f_1(\lambda) = \frac{\alpha_{12}^2}{\lambda\alpha_1\alpha_3} = \lambda^{-2}\frac{\alpha_2}{\alpha_3}\,\mathrm{CS}(\varepsilon_1, \varepsilon_2), \qquad f_2(\lambda) = \frac{\alpha_{13}^2}{\alpha_1\alpha_3} = \mathrm{CS}(\varepsilon_1, \varepsilon_3). \tag{3.31}$$

For the unique point $\lambda = \frac{\alpha_{12}}{\alpha_{13}} = \xi$ where $f_1(\lambda) = f_2(\lambda)$ we have

$$\xi^2 = \frac{\alpha_{12}^2}{\alpha_{13}^2} \leqslant \frac{\alpha_2}{\alpha_3}. \tag{3.32}$$

The case $\xi^2 = \frac{\alpha_2}{\alpha_3}$ means that $\mathrm{CS}(\varepsilon_1, \varepsilon_2) = \mathrm{CS}(\varepsilon_1, \varepsilon_3)$, for which for $\lambda^2 \leqslant \frac{\alpha_2}{\alpha_3}$ we have

$$f_1(\lambda) = \mathrm{CS}(\varepsilon_1, \varepsilon_2) \geqslant f_2(\lambda),$$

and for $\lambda^2 \geqslant \frac{\alpha_2}{\alpha_3}$ we have

$$f_2(\lambda) = \mathrm{CS}(\varepsilon_1, \varepsilon_3) \geqslant f_1(\lambda).$$

Thus $f = \max(f_1, f_2)$ has constant value $\mathrm{CS}(\varepsilon_1, \varepsilon_2) = \mathrm{CS}(\varepsilon_1, \varepsilon_3)$ on $[0, \infty]$.

We are left with the case

$$0 \leqslant \mathrm{CS}(\varepsilon_1, \varepsilon_2) < \mathrm{CS}(\varepsilon_1, \varepsilon_3).$$

Let $\sqrt{\frac{\alpha_2}{\alpha_3}}$ denote the square root of $\frac{\alpha_2}{\alpha_3}$ in the ordered abelian group $\mathcal{G}^{\frac{1}{2}} \supset \mathcal{G}$. We learn from (3.30) that $f_1(\lambda) \geqslant f_2(\lambda)$ if $0 \leqslant \lambda \leqslant \xi$, while $f_1(\lambda) \leqslant f_2(\lambda)$ if $\xi \leqslant \lambda \leqslant \infty$,[3] and thus obtain

$$f(\lambda) = \begin{cases} \mathrm{CS}(\varepsilon_1, \varepsilon_2) & 0 \leqslant \lambda \leqslant \xi, \\ \lambda^2 \frac{\alpha_3}{\alpha_2} \mathrm{CS}(\varepsilon_1, \varepsilon_3) & \xi \leqslant \lambda \leqslant \sqrt{\frac{\alpha_2}{\alpha_3}}, \\ \mathrm{CS}(\varepsilon_1, \varepsilon_3) & \sqrt{\frac{\alpha_2}{\alpha_3}} \leqslant \lambda \leqslant \infty. \end{cases} \tag{3.33}$$

We summarize all this as follows.

**Theorem 3.3.** *Assume that* $\alpha_{23}^2 \leqslant_\nu \alpha_2\alpha_3$ *and* $\mathrm{CS}(\varepsilon_1, \varepsilon_2) \leqslant \mathrm{CS}(\varepsilon_1, \varepsilon_3)$.

a) *If* $\mathrm{CS}(\varepsilon_1, \varepsilon_2) = \mathrm{CS}(\varepsilon_1, \varepsilon_3)$, *then* $f$ *is constant on* $[0, \infty]$ *with value* $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$.

b) *If* $\mathrm{CS}(\varepsilon_1, \varepsilon_2) < \mathrm{CS}(\varepsilon_1, \varepsilon_3)$, *then* $\xi < \sqrt{\frac{\alpha_2}{\alpha_3}}$. *If* $\sqrt{\frac{\alpha_2}{\alpha_3}} \in \mathcal{G}$, *then* $f$ *is constant with value* $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$ *on* $[0, \xi]$ *(in particular* $\xi = 0$ *if* $\mathrm{CS}(\varepsilon_1, \varepsilon_2) = 0$*), further increases strictly from* $\mathrm{CS}(\varepsilon_1, \varepsilon_2)$ *to* $\mathrm{CS}(\varepsilon_1, \varepsilon_3)$ *on* $\big[\xi, \sqrt{\frac{\alpha_2}{\alpha_3}}\big]$, *and then remains constant of value* $\mathrm{CS}(\varepsilon_1, \varepsilon_3)$ *on* $\big[\sqrt{\frac{\alpha_2}{\alpha_3}}, \infty\big]$. *If* $\sqrt{\frac{\alpha_2}{\alpha_3}} \notin \mathcal{G}$, *this holds again after replacing these intervals by* $\big[\xi, \sqrt{\frac{\alpha_2}{\alpha_3}}\big[ \;:= \big\{\lambda \in \mathcal{G} \mid \xi \leqslant \lambda < \sqrt{\frac{\alpha_2}{\alpha_3}}\big\}$ *and* $\big]\sqrt{\frac{\alpha_2}{\alpha_3}}, \infty\big] := \big\{\lambda \in \mathcal{G} \mid \sqrt{\frac{\alpha_2}{\alpha_3}} < \lambda < \infty\big\} \cup \{\infty\}$.

In the case $\sqrt{\frac{\alpha_2}{\alpha_3}} \in \mathcal{G}$ the graph of $f(\lambda)$ with respect to the variable $\lambda^2$ looks as follows.

## 4. The CS-profiles on a ray interval

As before we assume that $eR$ is a (bipotent) semifield. Given anisotropic rays $Y_1, Y_2, W$ in the $R$-module $V$ (i.e., $Y_1, Y_2, W \in \mathrm{Ray}(V_{\mathrm{an}})$, $V_{\mathrm{an}} := \{x \in V \mid q(x) \neq 0\} \cup \{0\}$) with $Y_1 \neq Y_2$), we are interested in the CS-*profile of* $W$ *on the interval* $[Y_1, Y_2]$, by which we mean the monotonicity behavior of the function $\mathrm{CS}(W, -)$ on $[Y_1, Y_2]$ with respect to the total ordering [4] $\leqslant_{Y_1}$, as studied in §3. (There we labeled $W = X_1$, $Y_1 = X_2$, $Y_2 = X_3$.)

More succinctly we denote the set $[Y_1, Y_2]$, equipped with the total ordering $\leqslant_{Y_1}$, by $\overrightarrow{[Y_1, Y_2]}$, and call it an **oriented closed ray interval**. Often we use the shorter term "$W$-**profile on** $\overrightarrow{[Y_1, Y_2]}$" instead of "CS-profile of $W$ on $\overrightarrow{[Y_1, Y_2]}$", whenever it is clear from the context that we are dealing with CS-ratios.

**Definition 4.1.** *Let* $W, Y_1, Y_2 \in \mathrm{Ray}(V_{\mathrm{an}})$ *be anisotropic, and assume that* $Y_1 \neq Y_2$.

[3] Actually we know this for long, cf. the arguments following (3.11).

[4] This also includes information about the zero set of this function.

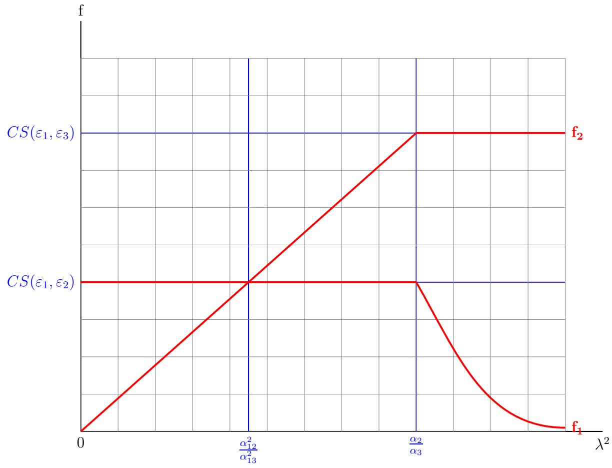


FIGURE 3. The case of $\sqrt{\frac{\alpha_3}{\alpha_3}} \in \mathcal{G}$.

a) *We call a $W$-profile on $\overrightarrow{[Y_1,Y_2]}$* ***ascending****, if* $\mathrm{CS}(W,Y_1) \leqslant \mathrm{CS}(W,Y_2)$, *and* ***descending*** *if* $\mathrm{CS}(W,Y_1) \geqslant \mathrm{CS}(W,Y_2)$.

b) *We say that a $W$-profile on $\overrightarrow{[Y_1,Y_2]}$ is* ***monotone****, if it is either increasing*[5] *or decreasing, and then usually speak of a "monotone $W$-profile on $[Y_1,Y_2]$", omitting the arrow indicating orientation, since it is irrelevant.*

c) *We say that a $W$-profile on $\overrightarrow{[Y_1,Y_2]}$ is* ***positive****, if* $\mathrm{CS}(W,Y_1) > 0$, $\mathrm{CS}(W,Y_2) > 0$, *and so* $\mathrm{CS}(W,Z) > 0$ *for all* $Z \in [Y_1,Y_2]$, *and we say that the $W$-profile is* ***non-positive*** *(or "attains zero"), if* $\mathrm{CS}(W,Y_1) = 0$ *or* $\mathrm{CS}(W,Y_2) = 0$.

**Remark 4.2.** *It is clear from §3 that if, say,* $\mathrm{CS}(W,Y_1) = 0$ *and* $\mathrm{CS}(W,Y_2) > 0$, *then* $\mathrm{CS}(W,Z) > 0$ *for all* $Z \neq Y_1$ *in* $[Y_1,Y_2]$, *while, if* $\mathrm{CS}(W,Y_1) = \mathrm{CS}(W,Y_2) = 0$, *then* $\mathrm{CS}(W,Z) = 0$ *for all* $Z \in [Y_1,Y_2]$.

We learned in §3 (cf. Theorems 3.2 and 3.3) that the monotonicity behavior of the function $\mathrm{CS}(W,-)$ on $\overrightarrow{[Y_1,Y_2]}$ is essentially determined[6] by the three ratios $\mathrm{CS}(Y_1,Y_2)$, $\mathrm{CS}(W,Y_1)$, $\mathrm{CS}(W,Y_2)$, more precisely by certain strict inequalities ($<$) and equalities ($=$) involving these CS-ratios. Accordingly, we classify the CS-profiles on $\overrightarrow{[Y_1,Y_2]}$ by characterizing them into "basic types", each is given by a conjunction $T$ of inequalities involving these CS-ratios and zero. From Theorems 3.2 and 3.3 we gain the following list of "**basic ascending types**", for which every ascending $W$-profile on $\overrightarrow{[Y_1,Y_2]}$ belongs to exactly one type $T$, and the condition $T$ encodes completely the monotonicity behavior of the function $\mathrm{CS}(W,-)$ on $[Y_1,Y_2]$.[7]

**Table 4.3.** *Assume that* $\mathrm{CS}(Y_1,Y_2) > e$.

[5]In basic terms, it means that $\mathrm{CS}(W,Z_1) \leqslant \mathrm{CS}(W,Z_2)$ for all $Z_1,Z_2 \in [Y_1,Y_2]$ with $[Y_1,Z_1] \subset [Y_1,Z_2]$.

[6]In the case $\mathrm{CS}(Y_1,Y_2) \leqslant e$ we also need the information whether the square class $eq(Y_1)q(Y_2) \subset \mathcal{G}$ of $eR$ is trivial or not (cf. Theorem 3.3).

[7]The reader may argue that our notion of basic type lacks a precise definition. We can remedy this by *defining* the basic types on $[Y_1,Y_2]$ as all the conditions $A$, $\partial A$, $B$, ... appearing in Tables 4.3, 4.4 and Scholium 4.5 below.

a) *The positive ascending basic types (i.e., types of positive ascending $W$-profiles) on* $\overrightarrow{[Y_1, Y_2]}$ *are*

$A$: $\quad 0 < \mathrm{CS}(W, Y_1) < \frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)}$,

$\partial A$: $\quad 0 < \mathrm{CS}(W, Y_1) = \frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)}$,

$B$: $\quad 0 < \frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)} < \mathrm{CS}(W, Y_1) < \mathrm{CS}(W, Y_2)$,

$\partial B$: $\quad 0 < \mathrm{CS}(W, Y_1) = \mathrm{CS}(W, Y_2)$.

*(Note that $\partial B$ implies* $0 < \frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)} < \mathrm{CS}(W, Y_1)$. *So we could also write*

$\partial B$: $\quad 0 < \frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)} < \mathrm{CS}(W, Y_1) = \mathrm{CS}(W, Y_2)$.*)*

b) *The non-positive ascending basic types are*

$E$: $\quad 0 = \mathrm{CS}(W, Y_1) < \mathrm{CS}(W, Y_2)$,

$\partial E$: $\quad 0 = \mathrm{CS}(W, Y_1) = \mathrm{CS}(W, Y_2)$.

*All these types are increasing, i.e., determine increasing $W$-profiles, except $B$ and $\partial B$.*

**Table 4.4.** *Assume that* $\mathrm{CS}(Y_1, Y_2) \leqslant e$ *and* $Y_1 \neq Y_2$. *We have the following list of ascending basic types on* $\overrightarrow{[Y_1, Y_2]}$.

a) *The positive ascending basic types*

$C$: $\quad 0 < \mathrm{CS}(W, Y_1) < \mathrm{CS}(W, Y_2)$,

$\partial C$: $\quad 0 < \mathrm{CS}(W, Y_1) = \mathrm{CS}(W, Y_2)$.

b) *The non-positive ascending basic types*[8]

$D$: $\quad 0 = \mathrm{CS}(W, Y_1) < \mathrm{CS}(W, Y_2)$,

$\partial D$: $\quad 0 = \mathrm{CS}(W, Y_1) = \mathrm{CS}(W, Y_2)$.

*Note that all these types are increasing.*

For each basic type $T$ there is a **reverse type** $T'$, obtained by interchanging $Y_1$ and $Y_2$ in condition $T$. Thus the reverses of the types listed in Tables 4.3 and 4.4 exhaust all basic types of descending CS-profiles on $\overrightarrow{[Y_1, Y_2]}$. We obtain the following list of such types.

**Scholium 4.5.**

a) *When* $\mathrm{CS}(Y_1, Y_2) > e$, *the descending basic types are* $A'$, $\partial A' := (\partial A)'$, $B'$, $\partial B' := (\partial B)' = \partial B$. $E'$, $\partial E' := (\partial E)' = \partial E$.

b) *When* $\mathrm{CS}(Y_1, Y_2) \leqslant e$, *the descending basic types are* $C'$, $\partial C' := \partial C$, $D'$, $\partial D' := (\partial D)' = \partial D$.

*All these types are decreasing except $B'$ and $\partial B'$, which are not monotone.*

In later sections additional conditions on $\mathrm{CS}(Y_1, Y_2)$, $\mathrm{CS}(W, Y_1)$, $\mathrm{CS}(W, Y_2)$ will come into play, which arise from a basic type $T$ by relaxing the strict inequality sign $<$ to $\leqslant$ at one or several places. We name such condition $U$ a **relaxation** of $T$, and call all the arising relaxations the **composed** CS-**types** on $[Y_1, Y_2]$. The reason for the latter term is that such a relaxation $U$ is a disjunction

$$U = T_1 \vee \cdots \vee T_r \tag{4.1}$$

of several basic types $T_i$, as will be seen (actually with $r \leqslant 4$). Since every $W$-profile on $\overrightarrow{[Y_1, Y_2]}$ belongs to exactly one basic type $T$, it is then obvious that the $T_i$ in (4.1) are uniquely

[8]Although $D$ and $\partial D$ are the same sentences as $E$ and $\partial E$ in Table 4.3, we use a different letter "$D$", since we include in the type the information whether $\mathrm{CS}(Y_1, Y_2) > e$ or $\mathrm{CS}(Y_1, Y_2) \leqslant e$.

determined by $U$ up to permutation, $T$ being one of them. We call the $T_i$ the **components** of the relaxation $U$.

We extend part of the terminology of basic types to their relaxations in the obvious way. The **reverse type** $U'$ of $U$ arises by interchanging $Y_1$ and $Y_2$ in the condition $U$. The composed type $U$ is **ascending** (resp. **descending**), if the sentence $\mathrm{CS}(W,Y_1) \leqslant \mathrm{CS}(W,Y_2)$ (resp. $\mathrm{CS}(W,Y_1) \geqslant \mathrm{CS}(W,Y_2)$) is a consequence of $U$, and $U$ is **positive**, if $U$ implies $0 < \mathrm{CS}(W,Y_i)$ for $i = 1,2$.

We list out all relaxations of all ascending basic types on $\overrightarrow{[Y_1,Y_2]}$, at first in the case $\mathrm{CS}(Y_1,Y_2) > e$, an then in the case $\mathrm{CS}(Y_1,Y_2) \leqslant e$. It will turn out that all these relaxations are again ascending.

**Scholium 4.6.** *Assume that* $\mathrm{CS}(Y_1,Y_2) > e$.

a) *The basic type* $E : 0 = \mathrm{CS}(W,Y_1) < \mathrm{CS}(W,Y_2)$ *has only one relaxation*

$$\overline{E}: \quad 0 = \mathrm{CS}(W,Y_1) \leqslant \mathrm{CS}(W,Y_2),$$

*for which* $\overline{E} = E \vee \partial E$.

b) *The basic type* $A : 0 < \mathrm{CS}(W,Y_1) < \frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)}$ *has the relaxations*

$$\begin{aligned} A_0: &\quad 0 \leqslant \mathrm{CS}(W,Y_1) < \tfrac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)}, \\ \overline{A}: &\quad 0 < \mathrm{CS}(W,Y_1) \leqslant \tfrac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)}, \\ \overline{A}_0: &\quad 0 \leqslant \mathrm{CS}(W,Y_1) \leqslant \tfrac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)}. \end{aligned}$$

*For these relaxations we have*

$$\begin{aligned} A_0 &= A \vee (0 = \mathrm{CS}(W,Y_1) < \mathrm{CS}(W,Y_2)) = A \vee E \\ \overline{A} &= A \vee (0 = \mathrm{CS}(W,Y_1) < \mathrm{CS}(W,Y_2)) = A \vee \partial A \\ \overline{A}_0 &= \overline{A} \vee (0 = \mathrm{CS}(W,Y_1) \leqslant \tfrac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)}) = \overline{A} \vee \overline{E} = A \vee \partial A \vee E \vee \partial E. \end{aligned}$$

c) *The positive relaxations of* $B : 0 < \frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)} < \mathrm{CS}(W,Y_1) < \mathrm{CS}(W,Y_2)$ *are*

$$\begin{aligned} \overline{B}: &\quad 0 < \tfrac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)} \leqslant \mathrm{CS}(W,Y_1) < \mathrm{CS}(W,Y_2), \\ \widetilde{B}: &\quad 0 < \tfrac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)} < \mathrm{CS}(W,Y_1) \leqslant \mathrm{CS}(W,Y_2), \\ \overline{\widetilde{B}}: &\quad 0 < \tfrac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)} \leqslant \mathrm{CS}(W,Y_1) \leqslant \mathrm{CS}(W,Y_2). \end{aligned}$$

*We have* $\overline{\widetilde{B}} = \overline{B} \vee \widetilde{B}$, *since* $\mathrm{CS}(W,Y_1) = \mathrm{CS}(W,Y_2)$ *implies* $\frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)} < \mathrm{CS}(W,Y_1)$, *and* $\frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)} = \mathrm{CS}(W,Y_1)$ *implies* $\mathrm{CS}(W,Y_1) < \mathrm{CS}(W,Y_2)$. *Also*

$$\widetilde{B} = B \vee (0 < \mathrm{CS}(W,Y_1) = \mathrm{CS}(W,Y_2)) = B \vee \partial B, \tag{4.2}$$

*since* $0 < \mathrm{CS}(W,Y_1) = \mathrm{CS}(W,Y_2)$ *implies* $0 < \frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(W,Y_2)} < \mathrm{CS}(W,Y_1)$, *and*

$$\overline{B} = B \vee (0 < \mathrm{CS}(W,Y_1) = \frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(W,Y_1)}) = B \vee \partial A. \tag{4.3}$$

*Finally*

$$\overline{\widetilde{B}} = \overline{B} \vee \widetilde{B} = B \vee \partial B \vee \partial A. \tag{4.4}$$

*We obtain the non-positive relaxations of $B$ by replacing in all these sentences $B, \overline{B}, \widetilde{B}, \widetilde{\overline{B}}$ the part $\left(0 < \frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)}\right)$ by $\left(0 \leqslant \frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)}\right) = \left(0 < \frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)}\right) \vee \big(0 = \mathrm{CS}(W,Y_2)\big)$. Since all 4 sentences $B, \overline{B}, \widetilde{B}, \widetilde{\overline{B}}$ have the consequence $\mathrm{CS}(W,Y_1) \leqslant \mathrm{CS}(W,Y_2)$, we get 4 non-positive relaxations of $B$, namely*

$$\begin{aligned} B_0 \ &= B \vee (0 = \mathrm{CS}(W,Y_1) = \mathrm{CS}(W,Y_2)) = B \vee \partial E, \\ \overline{B}_0 \ &= \overline{B} \cup \partial E, \\ \widetilde{B}_0 \ &= \widetilde{B} \vee \partial E, \\ \widetilde{\overline{B}}_0 \ &= \widetilde{B} \cup \partial E = B \vee \partial B \vee \partial A \vee \partial E. \end{aligned} \tag{4.5}$$

**Remark 4.7.** *Assume that* $\mathrm{CS}(Y_1, Y_2) > e$.

a) *We have*

$$\begin{aligned} A_0 \wedge \overline{A} \ &= (A \vee E) \wedge (A \vee \partial E) = A, \\ \overline{B} \wedge \widetilde{B} \ &= (B \vee \partial A) \wedge (B \vee \partial B) = B, \end{aligned}$$

*since different basic types are incompatible (= contradictory).*

b) *The condition*

$$\mathrm{Asc} := 0 \leqslant \mathrm{CS(W, Y_1)} \leqslant \mathrm{CS(W, Y_2)}$$

*is the disjunction of all ascending basic types,*

$$\mathrm{Asc} = \mathrm{A} \vee \partial\mathrm{A} \vee \mathrm{B} \vee \partial\mathrm{B} \vee \mathrm{E} \vee \partial\mathrm{E}, \tag{4.6}$$

*since every $W \in \mathrm{Ray}(V)$ fulfills exactly one of the basic type sentences.*
*The condition*

$$\mathrm{Asc}^+ := 0 < \mathrm{CS(W, Y_1)} < \mathrm{CS(W, Y_2)}$$

*is the disjunction of all positive strictly ascending basic types,*

$$\mathrm{Asc}^+ = \mathrm{A} \vee \partial\mathrm{A} \vee \mathrm{B}. \tag{4.7}$$

*Note that both* Asc *and* $\mathrm{Asc}^+$ *are not relaxations of basic types, and thus are not regarded as composite types.*

The table of composite types in the case $\mathrm{CS}(Y_1, Y_2) \leqslant e$ is much simpler.

**Scholium 4.8.** *Assume that* $\mathrm{CS}(Y_1, Y_2) \leqslant e$. *The basic type*

$$D : 0 = \mathrm{CS}(W, Y_1) < \mathrm{CS}(W, Y_2)$$

*has only one relaxation:*

$$\overline{D} = D \cup \partial D : 0 = \mathrm{CS}(W, Y_1) \leqslant \mathrm{CS}(W, Y_2).$$

*The basic type $C : 0 < \mathrm{CS}(W, Y_1) < \mathrm{CS}(W, Y_2)$ has three relaxations, namely*

$$\begin{aligned} \overline{C} \quad &:= \quad C \vee \partial C : 0 < \mathrm{CS}(W, Y_1) \leqslant \mathrm{CS}(W, Y_2), \\ C_0 \quad &:= \quad C \vee D : 0 \leqslant \mathrm{CS}(W, Y_1) < \mathrm{CS}(W, Y_2), \\ \overline{C}_0 \quad &:= \quad \overline{C} \vee D = \overline{C} \vee \overline{D} : 0 \leqslant \mathrm{CS}(W, Y_1) \leqslant \mathrm{CS}(W, Y_2). \end{aligned}$$

*$\overline{C}_0$ is the disjunction of all increasing basic types on $\overrightarrow{[Y_1, Y_2]}$.*

## 5. A convexity lemma for linear CS-inequalities, and first applications

As before we assume that $eR$ is a semifield and $(q, b)$ is a quadratic pair on an $R$-module $V$ with $q$ anisotropic.

**Lemma 5.1.** *Given $w_1, \dots, w_n \in V\setminus\{0\}$ and $\lambda_1, \dots, \lambda_n \in \mathcal{G}$, let $w := \sum_{i=1}^{n} \lambda_i w_i$. For any $y \in V\setminus\{0\}$ the following holds*

$$\mathrm{CS}(w, y) = \sum_{i=1}^{n} \mathrm{CS}(w_i, y)\alpha_i \tag{5.1}$$

*with $\alpha_i \in \mathcal{G}$, $0 < \alpha_i \leqslant e$, namely*

$$\alpha_i := \frac{q(\lambda_i w_i)}{q(w)}, \tag{5.2}$$

*and thus $0 < \alpha_i \leqslant e$.*

*Proof.* Since $b(w, y) = \sum_{i=1}^{n} b(\lambda_i w_i, y)$, we have

$$\begin{aligned}\mathrm{CS}(w, y) = \frac{b(w, y)^2}{q(w)q(y)} &= \sum_{i=1}^{n} \frac{b(\lambda_i w_i, y)^2}{q(w)q(y)} = \sum_{i=1}^{n} \frac{b(\lambda_i w_i, y)^2}{q(\lambda_i w_i)q(y)} \ \frac{q(\lambda_i w_i)}{q(w)} \\ &= \sum_{i=1}^{n} \mathrm{CS}(\lambda_i w_i, y)\frac{q(\lambda_i w_i)}{q(w)} = \sum_{i=1}^{n} \mathrm{CS}(w_i, y)\frac{q(\lambda_i w_i)}{q(w)}.\end{aligned}$$

□

**Lemma 5.2.** *Given $x_1, \dots, x_m$, $y_1, \dots, y_m$, $\alpha_1, \dots, \alpha_m$ in $eR$ such that that*

$$\alpha_i x_i < \alpha_i y_i \qquad \text{for } 1 \leqslant i \leqslant m, \tag{$*$}$$

*then*

$$\sum_{i=1}^{m} \alpha_i x_i < \sum_{i=1}^{m} \alpha_i y_i. \tag{$**$}$$

*Proof.* Choose $r \in \{1, \dots, m\}$ such that $\alpha_r x_r = \operatorname{Max}_{1 \leqslant i \leqslant m} \{\alpha_i x_i\}$, then

$$\sum_{i=1}^{m} \alpha_i x_i = \alpha_r x_r < \alpha_r y_r \leqslant \sum_{i=1}^{m} \alpha_i y_i.$$

□

**Remark 5.3.** *If $(*)$ holds with the strict inequality $<$ replaced by the weak inequality $\leqslant$ (respectively the equality sign $=$) everywhere, then $(**)$ holds with $\leqslant$ (respectively $=$) everywhere. This is trivial.*

We are ready to prove a convexity lemma for linear CS-inequalities, which will play a central role in the rest of the paper.

**Lemma 5.4** (CS-Convexity Lemma)**.** *Let $\square$ be one of the symbols $<, \leqslant, =$. Given rays $W_1, \dots, W_m$, $Y_1, \dots, Y_n$ in $V$ and scalars $\gamma_1, \dots, \gamma_n$, $\delta_1, \dots, \delta_n$ in $eR$ such that*

$$\sum_{j=1}^{n} \gamma_j \,\mathrm{CS}(W_i, Y_j) \ \square \ \sum_{j=1}^{n} \delta_j \,\mathrm{CS}(W_i, Y_j), \qquad \text{for } i = 1, \dots, m. \tag{$*$}$$

*Then, for every* $W \in \operatorname{conv}(W_1, \dots, W_m)$,

$$\sum_{j=1}^{n} \gamma_j \operatorname{CS}(W, Y_j) \ \square \ \sum_{j=1}^{n} \delta_i \operatorname{CS}(W, Y_j). \tag{$**$}$$

*Proof.* We verify the assertion for $<$. By Lemma 5.1 we have scalars $\alpha_1, \dots, \alpha_m \in \mathcal{G}$ such that

$$\operatorname{CS}(W, Y_j) = \sum_{i=1}^{m} \alpha_i \operatorname{CS}(W_i, Y_j)$$

for $j = 1, \dots, n$. Using Lemma 5.2 and the fact that the $\alpha_i$ are units of $eR$, we obtain

$$\begin{aligned}\sum_{j=1}^{n} \gamma_j \operatorname{CS}(W, Y_j) &= \sum_{j=1}^{n} \gamma_j \sum_{j=1}^{m} \alpha_i \operatorname{CS}(W_i, Y_j) = \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{n} \gamma_j \operatorname{CS}(W_i, Y_j) \\ &< \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{n} \delta_j \operatorname{CS}(W_i, Y_j) = \sum_{j=1}^{n} \delta_j \sum_{i=1}^{m} \alpha_i \operatorname{CS}(W_i, Y_j) = \sum_{j=1}^{n} \delta_j \operatorname{CS}(W, Y_j).\end{aligned}$$

For the other signs $\leqslant$, $=$ the argument is analogous, using Remark 5.3 instead of Lemma 5.2. (Here it does not matter that the $\alpha_i$'s are units of $eR$.) $\square$

We start with an application of the CS-Convexity Lemma 5.4 upon subsets of the ray space $\operatorname{Ray}(V)$ related to the CS-profile types introduced in §4.

**Definition 5.5.** *Given a pair* $(Y_1, Y_2)$ *of different rays in* $V$ *and a basic type* $T$ *(as listed in Tables 4.3, 4.4 and Scholium 4.5), we define the* $T$*-**locus of*** $\overrightarrow{[Y_1, Y_2]}$ *as the set of all rays* $W$ *in* $V$ *which have a* $W$*-profile of type* $T$ *on* $\overrightarrow{[Y_1, Y_2]}$*, and denote this subset of* $\operatorname{Ray}(V)$ *by* $\mathrm{Loc_T(Y_1, Y_2)}$.

It is understood that, if $T$ is defined under the condition, say, $\operatorname{CS}(Y_1, Y_2) > e$,[9] the sentence $(\operatorname{CS}(Y_1, Y_2) > e)$ is part of the sentence $T$, and thus $\mathrm{Loc_T(Y_1, Y_2)} = \varnothing$ when actually $\operatorname{CS}(Y_1, Y_2) \leqslant e$. So, using standard notation from logic, we may write

$$\mathrm{Loc_T(Y_1, Y_2) = \{W \in Ray(V) \mid W \models T\}} \tag{5.3}$$

for any pair $(Y_1, Y_2)$ of different rays in $V$ and any basic type $T$. We call these subsets $\mathrm{Loc_T(Y_1, Y_2)}$ of $\operatorname{Ray}(V)$ the **basic loci** of the interval $[Y_1, Y_2]$.

**Theorem 5.6.** *The family of nonempty basic loci of any interval* $[Y_1, Y_2]$ *in* $\operatorname{Ray}(V)$ *is a partition of* $\operatorname{Ray}(V)$ *into convex subsets.*

*Proof.* a) The family of basic loci of $[Y_1, Y_2]$ is a partition of $\operatorname{Ray}(V)$, as for a given $W \in \operatorname{Ray}(V)$ the function $\operatorname{CS}(W, -)$ on $\overrightarrow{[Y_1, Y_2]}$ has a profile of type $T$ for exactly one basic $T$.

b) Given a basic type $T$ for $\overrightarrow{[Y_1, Y_2]}$ it is a straightforward consequence of Lemma 5.4 (with $m = 2$) that $\mathrm{Loc_T(Y_1, Y_2)}$ is convex. We show this in the case that $\operatorname{CS}(Y_1, Y_2) > e$ and $T$ is the condition $B$ in Table 4.3. Let $W_1, W_2 \in \mathrm{Loc_B(Y_1, Y_2)}$ and $W \in [W_1, W_2]$ be given. Then

$$0 < \frac{\operatorname{CS}(W_i, Y_2)}{\operatorname{CS}(Y_1, Y_2)} < \operatorname{CS}(W_i, Y_1) < \operatorname{CS}(W_i, Y_2), \qquad \text{for } i = 1, 2. \tag{$*$}$$

[9]Taken up to interchanging $Y_1, Y_2$, the type $T$ is listed in Table 4.3.

Applying the CS-Convexity Lemma 5.4 to the inequality on the right (with $n = 1$, $\gamma_1 = \delta_1 = e$), we obtain

$$\mathrm{CS}(W, Y_1) < \mathrm{CS}(W, Y_2),$$

and thus also $0 < \frac{\mathrm{CS}(W,Y_2)}{\mathrm{CS}(Y_1,Y_2)}$. Applying the lemma to the inequality in the middle of $(*)$ (with $n = 1$, $\gamma_1 = \mathrm{CS}(Y_1, Y_2)^{-1}$, $\delta_1 = e$) we obtain

$$\frac{\mathrm{CS}(W, Y_2)}{\mathrm{CS}(Y_1, Y_2)} < \mathrm{CS}(W, Y_1).$$

This proves condition $B$ for $(W, Y_1, Y_2)$. □

**Definition 5.7.** *Given a composite type $U$ of CS-profiles (cf. §4), in analogy to* (5.3) *we define the $U$-**locus** of $\overrightarrow{[Y_1, Y_2]}$ as*

$$\mathrm{Loc_U(Y_1, Y_2) := \{W \in Ray(V) \mid W \models U\}}. \tag{5.4}$$

**Theorem 5.8.** *The subset* $\mathrm{Loc_U(Y_1, Y_2)}$ *is convex in* $\mathrm{Ray}(V)$ *for every pair* $(Y_1, Y_2)$ *of different rays in $V$.*

*Proof.* $U$ is by definition a relaxation of a basic type $T$, and so the inequalities in $U$ are obtained by replacing in $T$ the strict inequality $<$ by $\leqslant$ at several places. The CS-Convexity Lemma 5.4, taken now for weak inequalities, gives the claim. □

**Remark 5.9.** *As stated in* (4.1), *$U$ is a disjunction of finitely many basic types,*

$$U = T_1 \vee T_2 \vee \cdots \vee T_r.$$

*It follows from* (5.4) *that*

$$\mathrm{Loc_U(Y_1, Y_2) = \bigcup_{i=1}^{r} Loc_{T_i}(Y_1, Y_2)}. \tag{5.5}$$

## 6. Downsets of restricted QL-stars

Recall that the **QL-star** $\mathrm{QL}(X)$ of a ray $X$ (with respect to $q$) is the set of all $Y \in \mathrm{Ray}(V)$ for which the pair $(X, Y)$ is quasilinear; equivalently, the interval $[X, Y]$ is quasilinear [4, Definition 4.5]. The QL-stars determine the quasilinear behavior of $q$ on the ray space.

Given a QL-star $\mathrm{QL}(X)$ we investigate the **downset of** $\mathrm{QL}(X)$, i.e., the set of all QL-stars $\mathrm{QL}(Y) \subset \mathrm{QL}(X)$, partially ordered by inclusion. We translate this problem into the language of rays by considering the downset $\{Y \in \mathrm{Ray}(V) \mid Y \preceq_{\mathrm{QL}} X\}$ with respect to the quasiordering $\preceq_{\mathrm{QL}}$ given in (4.1). (Recall from [4, §5] that a QL-star $\mathrm{QL}(Y)$ corresponds uniquely to the equivalence class of $Y$ with respect to $\sim_{\mathrm{QL}}$.)

More generally fixing a nonempty set $D \subset \mathrm{Ray}(V)$, for any $X \in \mathrm{Ray}(V)$ we define

$$\mathrm{QL}_D(X) := \mathrm{QL}(X) \cap D,$$

and explore the downsets of this family of sets, ordered by inclusion. To do so, without extra costs, we pass to a coarsening $\preceq_D$ of the quasiordering $\preceq_{\mathrm{QL}}$, defined as

$$X \preceq_D X' \quad \Leftrightarrow \quad \mathrm{QL}_D(X) \subset \mathrm{QL}_D(X'),$$

with associated equivalence relation

$$X \sim_D X' \quad \Leftrightarrow \quad \mathrm{QL}_D(X) = \mathrm{QL}_D(X').$$

We call the set $\mathrm{QL}_D(X)$ the **restriction of the QL-star of $X$ to $D$.**

We use the monotone CS-profiles on an interval $[Y_1, Y_2]$, whenever they occur, to investigate these downsets. Yet, we need a criterion for quasilinearity of pairs of anisotropic rays in [7] (under a stronger assumption than before on $R$) which for the present paper reads as:

**Theorem 6.1** ([7, Theorems 6.7 and 6.11])**.** *Assume that $R$ is a **nontrivial tangible supersemifield**, i.e., $R$ is a supertropical semiring in which both $\mathcal{G} = eR\backslash\{0\}$ and $\mathcal{T} = R\backslash(eR)$ are abelian groups under multiplication, $e\mathcal{T} = \mathcal{G}$, and $\mathcal{G} \neq \{e\}$. Then a pair $(W, Z)$ of anisotropic rays on $V$ is quasilinear iff either $\mathrm{CS}(W, Z) \leqslant e$, or $\mathcal{G}$ is discrete, $\mathrm{CS}(W, Z) = c_0$[10], and both $W$ and $Z$ are g-isotropic (i.e., $q(w)$, $q(z) \in \mathcal{G}$ for all $w \in W$, $z \in Z$), saying in the latter case that $(W, Z)$ is **exotic quasilinear**.*

**Theorem 6.2.** *Assume that $R$ is a nontrivial tangible supersemifield and that the quadratic form $q$ is anisotropic on $V$. Given a (nonempty) subset $D$ of $\mathrm{Ray}(V)$ let $X, Y_1, Y_2$ be rays in $V$ with $Y_1 \preceq_D X$ and $Y_2 \preceq_D X$, and assume that the CS-profile of every $W \in D$ on $[Y_1, Y_2]$ is monotone. Then the following holds.*

i) *If $\mathcal{G}$ is dense, then $Y \preceq_D X$ for every $Y \in [Y_1, Y_2]$.*
ii) *If $\mathcal{G}$ is discrete, then $Y \preceq_D X$ for every g-anisotropic $Y \in [Y_1, Y_2]$.*
iii) *If $\mathcal{G}$ is discrete and at least one of the rays $Y_1, Y_2$ is g-isotropic, then $Y \preceq_D X$ for every $Y \in [Y_1, Y_2]$.*

*Proof.* The study in §3 of the functions $\mathrm{CS}(W, -)$ on closed intervals reveals that for a given $W \in \mathrm{Ray}(V)$ this function is monotone, i.e., increasing or decreasing on $\overrightarrow{[Y_1, Y_2]}$ iff

$$\forall Z \in [Y_1, Y_2] : \quad \mathrm{CS}(W, Z) \geqslant \min(\mathrm{CS}(W, Y_1), \mathrm{CS}(W, Y_2)). \tag{6.1}$$

In the following we only rely on this property.

Let $Y \in [Y_1, Y_2]$ and $W \in \mathrm{QL}(Y) \cap D$ be given. We prove that $W \in \mathrm{QL}(X)$ under conditions i) – iii), and then will be done.

a) Suppose that $\mathrm{CS}(W, Y) \leqslant e$. Since the $W$-profile on $[Y_1, Y_2]$ is monotonic, we conclude from (6.1) that $\mathrm{CS}(W, Y_i) \leqslant e$ for $i = 1$ or 2, whence $W \in \mathrm{QL}_D(Y_i)$, and so $W \in \mathrm{QL}_D(X)$ since $Y_i \preceq_D X$. This settles claim i) of the theorem, as well as the other claims in the case $\mathrm{CS}(W, Y) \leqslant e$.

b) There remains the case that $eR$ is discrete and $\mathrm{CS}(W, Y) = c_0$. The pair $(W, Y)$ is exotic quasilinear, since $W \in \mathrm{QL}(Y)$, and thus $W$ and $Y$ are $g$-isotropic. But, under the assumption in ii) of the theorem this does not hold, whence this case cannot occur.

c) We are left with a proof of part iii). If $\mathrm{CS}(W, Y_1) \leqslant e$ or $\mathrm{CS}(W, Y_2) \leqslant e$, the same argument as in a) gives that $W \in \mathrm{QL}(Y_1)$ or $W \in \mathrm{QL}(Y_2)$, and so $W \in \mathrm{QL}(X)$. Henceforth we assume that $\mathrm{CS}(W, Y_1) = \mathrm{CS}(W, Y_2) = c_0$. If, say, $Y_1$ is $g$-isotropic, then the pair $(W, Y_1)$ is exotic quasilinear, whence $W \in \mathrm{QL}(Y_1)$, and so $W \in \mathrm{QL}(X)$, as desired. □

We list several cases where a given CS-profile is monotonic, now using in detail the profile analysis from §3. (Here it suffices to assume that $eR$ is a semifield.)

**Scholium 6.3.** *As before we assume that $q$ is anisotropic on $V$.*

[10] $c_0$ denotes the smallest element $> e$ in $\mathcal{G}$. It exists since $\mathcal{G}$ is discrete.

a) *If* $[Y_1, Y_2]$ *is* $\nu$*-quasilinear, then* $[Y_1, Y_2]$ *has a monotonic* $W$*-profile for every* $W \in \mathrm{Ray}(V)$.

b) *If* $[Y_1, Y_2]$ *is* $\nu$*-excessive, with critical rays* $Y_{12}$ *(near* $Y_1$*) and* $Y_{21}$ *(near* $Y_2$*), then both* $[Y_1, Y_{12}]$ *and* $[Y_{21}, Y_2]$ *have a monotonic* $W$*-profile for every* $W$.

c) *Given* $W \in \mathrm{Ray}(V)$ *let* $M = M(W, Y_1, Y_2)$ *denote the* $W$*-median of* $[Y_1, Y_2]$*. Assume that the* $W$*-profile of* $[Y_1, Y_2]$ *is not monotone. Then* $[Y_1, M]$ *and* $[M, Y_2]$ *are the maximal closed subintervals of* $[Y_1, Y_2]$ *with a monotone* $W$*-profile.*

d) *Of course, if* $[Y_1, Y_2]$ *has a monotonic* $W$*-profile for a given ray* $W$*, then the same holds for every closed subinterval of* $[Y_1, Y_2]$.

We next search for rays $Z \preceq_D Y$ in a given interval $[X, Y]$. Here and elsewhere it is convenient to extend our notion of QL-stars and related objects from rays to vectors in a trivial way as follows (assuming only that $eR$ is a semifield).

**Notation 6.4.** *Given* $x, y \in V \setminus \{0\}$ *we define*

$$\mathrm{QL}(x) := \mathrm{QL}(\mathrm{ray}(x)),$$

*and set*

$$x \preceq_{\mathrm{QL}} y \quad \Leftrightarrow \quad \mathrm{ray}(x) \preceq_{\mathrm{QL}} \mathrm{ray}(y),$$

*equivalently*

$$x \preceq_{\mathrm{QL}} y \quad \Leftrightarrow \quad \mathrm{QL}(x) \subset \mathrm{QL}(y).$$

*Consequently we define*

$$x \sim_{\mathrm{QL}} y \quad \Leftrightarrow \quad \mathrm{QL}(x) = \mathrm{QL}(y) \quad \Leftrightarrow \quad \mathrm{ray}(x) \sim_{\mathrm{QL}} \mathrm{ray}(y).$$

*More generally, if a set* $D \subset \mathrm{Ray}(V)$ *is given, we put*

$$\begin{aligned} \mathrm{QL}_D(x) &= \mathrm{QL}_D(\mathrm{ray}(x)), \\ x \preceq_D y &\Leftrightarrow \mathrm{QL}_D(x) \subset \mathrm{QL}_D(y), \\ x \sim_D y &\Leftrightarrow \mathrm{QL}_D(x) = \mathrm{QL}_D(y). \end{aligned}$$

As before we assume that $q$ is anisotropic on $V$, and that $R$ is a nontrivial tangible supersemifield.

**Lemma 6.5.** *Let* $x, y \in V \setminus \{0\}$ *and assume that* $q(x+y) \cong_\nu q(y)$.

a) *For any* $w \in V \setminus \{0\}$

$$\mathrm{CS}(w, y) \leqslant \mathrm{CS}(w, x+y).$$

b) *If* $q(y) \in \mathcal{G}$*, then* $q(x+y) = q(y)$.

*Proof.* a): As in the proof of Lemma 5.1, we have

$$\mathrm{CS}(w, x+y) = \mathrm{CS}(w, x)\frac{q(x)}{q(x+y)} + \mathrm{CS}(w, y)\frac{q(y)}{q(x+y)}$$

which implies that $\mathrm{CS}(w, y) \leqslant \mathrm{CS}(w, x+y)$, since $\frac{q(y)}{q(x+y)} \cong_\nu e$.

b): A priori we have $q(x+y) = q(x) + q(y) + b(x,y)$, and so $q(y) \leqslant q(x+y)$ in the minimal ordering on $R$. Since by assumption $eq(y) = eq(x+y)$ and $q(y) \in \mathcal{G}$, it follows that $q(x+y) = q(y)$.[11] □

**Theorem 6.6.** *Assume that $q(x+y) \cong_\nu q(y)$ for given $x, y \in V \backslash \{0\}$. When $\mathcal{G}$ is discrete, assume also that $q(y) \in \mathcal{G}$.*

a) $x + y \preceq_{\mathrm{QL}} y$ *(and so $x + y \preceq_D y$ for any $D \subset \mathrm{Ray}(V)$).*

b) *The interval* $[\mathrm{ray}(x+y), \mathrm{ray}(y)]$ *has a monotone $W$-profile for every ray $W$ in $V$.*

*Proof.* a): Let $w \in V \backslash \{0\}$ be given with $w \in \mathrm{QL}(x+y)$, we verify that $w \in \mathrm{QL}(y)$. By Lemma 6.5.a we know that

$$\mathrm{CS}(w, y) \leqslant \mathrm{CS}(w, x+y). \tag{$*$}$$

Assume first that $\mathcal{G}$ is dense, then $\mathrm{CS}(w, x+y) \leqslant e$, since $w \in \mathrm{QL}(x+y)$. From $(*)$ it follows that $\mathrm{CS}(w, y) \leqslant e$, and so $w \in \mathrm{QL}(y)$.

Assume next that $\mathcal{G}$ is discrete. If $\mathrm{CS}(w, y) \leqslant e$, then certainly $w \in \mathrm{QL}(y)$. There remains the case that

$$\mathrm{CS}(w, y) \geqslant c_0. \tag{$**$}$$

Now $(*)$ tells us that $\mathrm{CS}(w, x+y) \geqslant c_0$. Since $w \in \mathrm{QL}(x+y)$, we conclude by Theorem 6.1 that $\mathrm{CS}(w, x+y) = c_0$ and $q(w) \in \mathcal{G}$, $q(x+y) \in \mathcal{G}$. From $(*)$ and $(**)$ we infer that $\mathrm{CS}(w, y) = c_0$. Thus, again by Theorem 6.1, $w \in \mathrm{QL}(y)$.

b): Let $Z := \mathrm{ray}(x+y)$, $Y := \mathrm{ray}(y)$. We proved that $Z \preceq_{\mathrm{QL}} Y$, i.e., $\mathrm{QL}(Z) \subset \mathrm{QL}(Y)$. This implies that $Z \in \mathrm{QL}(Y)$, i.e., that $[Y, Z]$ is quasilinear. All the more $[Y, Z]$ is $\nu$-quasilinear, and Scholium 6.3.a confirms that it has a monotone $W$-profile for any $W \in \mathrm{Ray}(V)$. □

## 7. The medians of a closed ray-interval

For a short period we only assume that $(q, b)$ is a quadratic pair on an $R$-module $V$ where $R$ is a supertropical semiring without zero divisors, and $\lambda x \neq 0$ for nonzero $\lambda \in R$ and all nonzero $x \in V$, cf. [7, §6]. Recall that two vectors $x, x' \in V$ are said to be **ray-equivalent**, written $x \sim_r x'$, if there exist scalars $\lambda, \lambda' \in R \backslash \{0\} = \mathcal{G}$ with $\lambda x = \lambda' x'$; the ray-equivalence class of $x \neq 0$ is denoted $\mathrm{ray}(x)$. We introduce a map

$$m : V \times V \times V \longrightarrow V$$

by the rule

$$m(w, x, y) := b(w, y)x + b(w, x)y. \tag{7.1}$$

Obviously this map is $R$-trilinear, and so is compatible with ray-equivalence, i.e., if $w \sim_r w'$, $x \sim_r x'$, $y \sim_r y'$ then $m(w, x, y) \sim_r m(w', x', y')$. Thus for any three rays $W = \mathrm{ray}(w)$, $X = \mathrm{ray}(x)$, $Y = \mathrm{ray}(y)$, where **not** $b(w, x) = b(w, y) = 0$, we obtain a well defined ray

$$M(W, X, Y) = \mathrm{ray}(m(w, x, y)) \in [X, Y]. \tag{7.2}$$

In fact, at least one of the vectors $b(w, y)x$, $b(w, x)y$ is not zero, and so

$$m(w, x, y) \in (Rx + Ry) \backslash \{0\}.$$

Here the notion of the "polar" of a subset $C$ of $\mathrm{Ray}(V)$ comes into play, defined as follows.

[11] Note that $\alpha \cong_\nu \beta$, $\alpha \in \mathcal{G} \Rightarrow \beta \leqslant \alpha$ for any $\alpha, \beta \in R$.

**Definition 7.1.** *Given a (nonempty) subset $C$ of* $\mathrm{Ray}(V)$*, the* ***polar*** $C^\perp$ *is the set of all* $W \in \mathrm{Ray}(V)$ *with* $b(w,x) = 0$ *for all* $w \in W$ *and* $x \in V\backslash\{0\}$ *with* $\mathrm{ray}(x) \in C$.

It is immediate from Definition 7.1 that any polar $C^\perp$ is convex and also that a set $C \subset \mathrm{Ray}(V)$ and its convex hull $\mathrm{conv}(C)$ have the same polar,

$$C^\perp = \mathrm{conv}(C)^\perp. \tag{7.3}$$

Thus it suffices most often to consider polars of convex sets. If $C$ is convex, then we can characterize both $C$ and $C^\perp$ by the ray closed subsets $\widetilde{C}$ and $(C^\perp)^\sim$ of $V\backslash\{0\}$, associated to $C$ and $C^\perp$ (cf. [4, Notation 2.4]) apparently as follows:

$$(C^\perp)^\sim = \{w \in V\backslash\{0\} \mid b(w,x) = 0 \text{ for every } x \in \widetilde{C}\}. \tag{7.4}$$

For the ray-closed submodules

$$T = \widetilde{C} \cup \{0\}, \quad U = (C^\perp)^\sim \cup \{0\}$$

(cf. [4, Remark 2.6]) we conclude that

$$U = \{w \in V \mid b(w,t) = 0 \text{ for every } t \in T\}. \tag{7.5}$$

We now take a look at the complement of a polar in $\mathrm{Ray}(V)$.

**Proposition 7.2.** *Assume that $C$ is a subset of* $\mathrm{Ray}(V)$ *and that $W_1$ is a ray in $V$ where $W_1 \notin C^\perp$. Then for any $W_2 \in \mathrm{Ray}(V)$ the half-open interval $[W_1, W_2[$ [12] is disjoint from $C^\perp$.*

*Proof.* Writing $X^\perp := \{X\}^\perp$ for $X \in \mathrm{Ray}(V)$ it is obvious that

$$C^\perp = \bigcap_{X \in C} X^\perp. \tag{7.6}$$

Thus, there exists some $X \in C$ such that $W_1 \notin X^\perp$. Picking vectors $w_1 \in W_1$, $w_2 \in W_2$, $x \in X$, then $b(w_1,x) \neq 0$, which implies that for any scalars $\lambda_1 \in R\backslash\{0\}$, $\lambda_2 \in R$

$$b(\lambda_1 w_1 + \lambda_2 w_2, x) = \lambda_1 b(w_1,x) + \lambda_2 b(w_2,x) \neq 0.$$

Since such vectors $\lambda_1 w_1 + \lambda_2 w_2$ represent all rays in $[W_1, W_2[$ , we conclude that $[W_1, W_2[ \cap X^\perp = \emptyset$. All the more $[W_1, W_2[ \cap C^\perp = \emptyset$. ☐

**Corollary 7.3.** *Given any subset $C$ of* $\mathrm{Ray}(V)$*, both sets $C^\perp$ and* $\mathrm{Ray}(V)\backslash C^\perp$ *are convex.*

*Proof.* Convexity of $C^\perp$ had been observed above. The convexity of $\mathrm{Ray}(V)\backslash C^\perp$ follows from Proposition 7.2 by taking $W_2$ in $\mathrm{Ray}(V)\backslash C^\perp$. ☐

We are ready for the key definition of this section.

**Definition 7.4.** *Given three rays $W, X, Y$ in $V$ with $W \notin [X,Y]^\perp = \{X,Y\}^\perp$, the ray $M(W,X,Y)$ from* (7.2) *is called the $W$-**median** of the pair $(X,Y)$.*

We denote this ray most often by $M_W(X,Y)$ instead of $M(W,X,Y)$. This notation emphasizes the fact, obvious from (7.1) and (7.2), that

$$M_W(X,Y) = M_W(Y,X). \tag{7.7}$$

The assignment $W \mapsto M_W(X,Y)$ has convexity properties as follows.

[12] The overall assumption that $eR$ is a semifield is not necessary, cf. [7, Definition 7.5].

**Theorem 7.5.** *Assume that $W_1, W_2, X, Y$ are rays in $V$ with $W_1 \notin \{X,Y\}^\perp$, $W_2 \notin \{X,Y\}^\perp$.*

a) $[W_1, W_2] \cap \{X,Y\}^\perp = \varnothing$ *and so $M_W(X,Y)$ is defined for every $W \in [W_1, W_2]$.*

b) *For any $W \in [W_1, W_2]$*

$$M_W(X,Y) \in [M_{W_1}(X,Y), M_{W_2}(X,Y)]. \tag{7.8}$$

*Proof.* a): Clear from Proposition 10.2, applied to $C = \{X,Y\}$.

b): We may assume that $W \neq W_1$, $W \neq W_2$. For vectors $w_1 \in W_1$, $w_2 \in W_2$, $x \in X$, $y \in Y$. there exist scalars $\lambda_1, \lambda_2 \in R\backslash\{0\}$ such that $w = \lambda_1 w_1 + \lambda_2 w_2 \in W$. Then $m(w,x,y) = \lambda_1 m(w_1,x,y) + \lambda_2 m(w_2,x,y)$, and so $M_W(X,Y) = \mathrm{ray}(\lambda_1 m(w_1,x,y) + \lambda_2 m(w_2,x,y)) \in [M_{W_1}(X,Y), M_{W_2}(X,Y))]$. □

We state an immediate consequence of this theorem.

**Corollary 7.6.** *Let $X, Y$ be (different) rays in $V$. Assume that $S$ is a convex subset of the closed interval $[X,Y]$. Then the set of all rays $W$ in $V$ with $W \notin \{X,Y\}^\perp$ and $M_W(X,Y) \in S$ is convex in* $\mathrm{Ray}(V)$.

Assume now again, as mostly from §6 onward, that $eR$ is a semifield. Then we know from [7, Theorem 8.8], that the "border rays" $X, Y$ of $[X,Y]$ are uniquely determined by $[X,Y]$ up to permutation. Thus we are entitled to call $M_W(X,Y)$ the $W$**-median of** $[X,Y]$.

The $W$-median $M_W(X,Y)$ have already appeared in our analysis of the function $f(\lambda) = CS(\varepsilon_1, \varepsilon_2 + \lambda\varepsilon_3)$ in §3 under the labeling $X_1, X_2, X_3$ instead of $W, X, Y$, with $\varepsilon_i \in X_i$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ instead of $w, x, y$. From the analysis in §3 we can read off the following important facts about $M_W(X,Y)$.

**Theorem 7.7.** *Assume that $eR$ is a semifield and $W, X, Y$ are rays in $V$ where $W \notin [X,Y]^\perp = \{X,Y\}^\perp$, so that the $W$-median $M_W(X,Y)$ is defined. Assume also that the rays $W, X, Y$ are anisotropic (i.e., none of the sets $q(W)$, $q(X)$, $q(Y)$ contains zero), so that the* CS*-ratio* $\mathrm{CS}(W,Z)$ *is defined for every $Z \in [X,Y]$.*

a) *The function*

$$Z \longmapsto \mathrm{CS}(W,Z), \quad [X,Y] \longrightarrow eR$$

*attains its minimal value at*

$$M := M_W(X,Y).$$

b) *If the function* $\mathrm{CS}(W,-)$ *is monotone on $[X,Y]$ (with respect to $\leqslant_X$), then*

$$\mathrm{CS}(W,M) = \min(\mathrm{CS}(W,X), \mathrm{CS}(W,Y)) \tag{7.9}$$

*is this minimal value.*

c) $\mathrm{CS}(W,-)$ *is monotone on $[X,Y]$ iff either* $\mathrm{CS}(X,Y) \leqslant e$ *or* $\mathrm{CS}(X,Y) > e$ *and $M \notin ]X^*, Y^*[$ where $X^*, Y^*$ denote the critical rays of $[X,Y]$ near $X$ and $Y$ respectively (cf.* [7, §9]*).*

d) *If* $\mathrm{CS}(W,-)$ *is not monotone (and so* $\mathrm{CS}(X,Y) > e$*, $M \in ]X^*, Y^*[$ ), the minimal value of* $\mathrm{CS}(W,-)$ *on $[X,Y]$ is*

$$\mathrm{CS}(W,M) = \sqrt{\frac{\mathrm{CS}(W,X)\,\mathrm{CS}(W,Y)}{\mathrm{CS}(X,Y)}} < \min(\mathrm{CS}(W,X), \mathrm{CS}(W,Y)). \tag{7.10}$$

*Furthermore, $Z = M$ is the* ***only ray in*** $[X, Y]$ ***where the minimum is attained.***

*Proof.* As pointed out, all proofs have been done in §3. The ray $M$ corresponds to $\lambda = \xi := \frac{\alpha_{12}}{\alpha_{13}}$, cf. (3.11) and (3.12). Claim a) is evident by the argument following (3.11). Claim b) then is trivial. The other two claims c) and d) follow from the description of the monotonic behavior of $f(\lambda)$ in Proposition 3.1 and Theorems 3.2 and 3.3. Note that there, when $\mathrm{CS}(X_2, X_3) > e$, the critical rays of $[X_2, X_3]$ near $X_2$ and $X_3$ (as defined in [7, §9]) correspond to $\lambda = \frac{\alpha_2}{\alpha_{23}}$ and $\lambda = \frac{\alpha_{23}}{\alpha_3}$ respectively, cf. (3.20). $\square$

**Remark 7.8.** *In the terminology of §4 the function* $\mathrm{CS}(W, -)$ *on* $[X, Y]$ *is not monotone iff the* CS*-profile of* $W$ *on* $[X, Y]$ *is of type* $B, B'$ *or* $\partial B$ $(= \partial B')$*, cf. Tables 4.3 and 4.4 and Scholium 4.5. The ray* $W$ *is in the polar* $\{X, Y\}^{\perp}$ *iff* $\mathrm{CS}(W, -)$ *is zero everywhere on* $[X, Y]$*, which means that the* CS*-profile of* $W$ *on* $[X, Y]$ *is of type* $\partial D$ *or* $\partial E$.

## 8. On maxima and minima of $\mathrm{CS}(W, -)$ on finitely generated convex sets in the ray space

We call a convex subset $C$ of $\mathrm{Ray}(V)$ **finitely generated**, if $C$ is the convex hull of a finite set of rays $\{Y_1, \dots, Y_n\}$, and call $\{Y_1, \dots, Y_n\}$ a **set of generators** of $C$. Note that then the sum $(Y_i)_0 + \cdots + (Y_n)_0$ of the submodules $(Y_i)_0 = Y_i \cup \{0\}$ of $V$ is the ray-closed submodule $U$ of $V$ with $\mathrm{Ray}(U) = C$, cf. §2.

Assume again, as previously, that the ghost ideal $eR$ of the supertropical semiring $R$ is a semifield and $(q, b)$ is a quadratic pair on the $R$-module $V$. Assume also that $q$ is anisotropic on $V$. (Otherwise we replace $V$ by $V_{\mathrm{an}}$). Then the CS-ratio $\mathrm{CS}(W, Z)$ is defined for any two rays $W, Z$ in $V$. Assume finally that $C$ is a finitely generated convex subset of $\mathrm{Ray}(V)$ and $(Y_1, \dots, Y_n)$ is a fixed sequence of generators of $C$.

Given a ray $W$ on $V$, we enquire whether the function $\mathrm{CS}(W, -)$ on $C$ has a minimal value, and then, where on $C$ this minimal value is attained.[13] To have a precise hold at the function $Y \mapsto \mathrm{CS}(W, Y)$, $C \to eR$, we use the following notation. Given vectors $y_i \in eY_i$ $(1 \leqslant i \leqslant n)$, a ray $W \in \mathrm{Ray}(V)$, and vectors $w \in eW$, $y \in eY$, $Y \in C$, then $Y_i = \mathrm{ray}(y_i)$ and $W = \mathrm{ray}(w)$, $Y = \mathrm{ray}(y)$. We have a presentation

$$y = \lambda_1 y_1 + \cdots + \lambda_n y_n \tag{8.1}$$

with a sequence $(\lambda_1, \dots, \lambda_n) \in eR^n$, not all $\lambda_i = 0$.

Let $\alpha_i := q(y_i)$, $\beta_{ij} := b(y_i, y_j)$. For $i, j \in \{1, \dots, n\}$ we define

$$\gamma_{ij} = \begin{cases} \alpha_i & \text{if} \quad i = j, \\ \beta_{ij} & \text{if} \quad i \neq j, \end{cases}$$

and write

$$q(y) = \sum_{1 \leqslant i \leqslant j \leqslant n} \gamma_{ij} \lambda_i \lambda_j. \tag{8.2}$$

[13]We leave the important problem aside, whether $C$ has a *unique minimal* set of generators. It would take us too far afield.

By a computation as in the proof of Lemma 5.1 we obtain a useful formula for $\mathrm{CS}(W,Y) = \mathrm{CS}(w,y)$, interchanging there the arguments $w, y$, namely

$$\mathrm{CS}(w,y) = \sum_{i=1}^{n} \mathrm{CS}(w, \lambda_i y_i) \frac{q(\lambda_i y_i)}{q(y)},$$

and so

$$\mathrm{CS}(W,Y) = \sum_{i=1}^{n} \mathrm{CS}(W, Y_i) \frac{\lambda_i^2 \alpha_i}{q(y)}. \tag{8.3}$$

We now are ready for the central result of this section.

**Theorem 8.1.** *Let $W, Y_1, \dots, Y_n \in \mathrm{Ray}(V)$ be given and $C := \mathrm{conv}(Y_1, \dots, Y_n)$. Then the $eR$-valued function $\mathrm{CS}(W,-)$ on $C$ has a minimum. It is attained at $Y_r$ for some $r \in \{1,\dots,n\}$ or at the $W$-median $M_W(Y_r, Y_s)$ of some interval $[Y_r, Y_s]$, $1 \leqslant r \leqslant s \leqslant n$, on which $\mathrm{CS}(W,-)$ is not monotone.*

*Proof.* Without loss of generality we assume that $\mathrm{CS}(W,Y_1) \leqslant \mathrm{CS}(W,Y_i)$ for $2 \leqslant i \leqslant n$. We distinguish two cases.

**Case A**: $\mathrm{CS}(W,Y_1) \leqslant \mathrm{CS}(W,Y)$ for every $Y \in C$. Now the minimum is attained at $Y_1$.

**Case B**: There exists some $Y \in C$ with $\mathrm{CS}(W,Y) < \mathrm{CS}(W,Y_1)$.

We choose $(\lambda_1, \dots, \lambda_n) \in eR^n$ such that (8.1) and (8.2) hold for $W$ and $Y$, and so does (8.3). We further choose a dominant term $\gamma_{r,s}\lambda_r\lambda_s$, $r \leqslant s$, in the sum on the right of (8.2). Then $q(y) = \gamma_{rs}\lambda_r\lambda_s$, and so

$$\mathrm{CS}(W,Y) = \sum_{i=1}^{n} \mathrm{CS}(W, Y_i) \frac{\lambda_i^2 \alpha_i}{\gamma_{rs}\lambda_r\lambda_s}.$$

Clearly

$$\sum_{i=r}^{s} \mathrm{CS}(W, Y_i) \frac{\alpha_i \lambda_i^2}{\gamma_{rs}\lambda_r\lambda_s} \leqslant \mathrm{CS}(W,Y) < \mathrm{CS}(W,Y_1). \tag{$*$}$$

Suppose that $r = s$. Then we obtain that $\mathrm{CS}(W,Y_r) < \mathrm{CS}(W,Y_1)$, contradicting our initial assumption that $\mathrm{CS}(W,Y_1) \leqslant \mathrm{CS}(W,Y_i)$ for all $i \in \{1,\dots,n\}$. Thus $r < s$. Let

$$Z := \mathrm{ray}(\lambda_r y_r + \lambda_s y_s) \in [Y_r, Y_s].$$

Formula (8.3) for this ray $Z$ tells us that the sum on the left in $(*)$ equals $\mathrm{CS}(W,Z)$. Thus

$$\mathrm{CS}(W,Z) \leqslant \mathrm{CS}(W,Y) < \mathrm{CS}(W,Y_1).$$

It follows that $\mathrm{CS}(W,Z)$ is smaller than both $\mathrm{CS}(W,Y_r)$ and $\mathrm{CS}(W,Y_s)$. By our analysis of the minimal value of $\mathrm{CS}(W,-)$ on closed intervals (Theorem 7.7), we conclude that $\mathrm{CS}(W,-)$ is not monotone on $[Y_r, Y_s]$. So, the minimum of $\mathrm{CS}(W,-)$ on $[Y_r, Y_s]$ is attained at the $W$-median $M_W(Y_r, Y_s)$ and only at this ray. It is now evident that the minimum of $\mathrm{CS}(W,-)$ on $C$ exists and is attained at one of the finitely many $W$-medians $M_W(Y_i, Y_j)$ with $\mathrm{CS}(W,-)$ not monotone on $[Y_i, Y_j]$. $\square$

Given a finitely generated convex set $C$ in $\mathrm{Ray}(V)$, a sequence $(Y_1, \dots, Y_n)$ of generators of $C$, and a ray $W$ on $V$, we define

$$\mu_W(Y_1, \dots, Y_n) := \mu_W(C) := \min_{Z \in C} \mathrm{CS}(W, Z). \tag{8.4}$$

**Theorem 8.2.** *In this setting assume for a fixed $W$ that*

$$\mu_W(C) < \min_{1 \leqslant i \leqslant n} \mathrm{CS}(W, Y_i),$$

*that $Y \in C$ is a ray with $\mathrm{CS}(W,Y) = \mu_W(C)$, and that a presentation $Y = \mathrm{ray}(\lambda_1 y_1 + \cdots + \lambda_n y_n)$ is given with $y_i \in eY_i$, $\lambda_i \in eR$. Then for every dominant term $\gamma_{rs}\lambda_r\lambda_s$ in the sum* (8.2) *above we have $r < s$. The ray*

$$Z_{rs} := \mathrm{ray}(\lambda_r y_r + \lambda_s y_s) \in [Y_r, Y_s]$$

*is the $W$-median of $[Y_r, Y_s]$, and*

$$\mu_W(C) = \mu_W(Y_r, Y_s) = \mathrm{CS}(W, Z_{rs}) = \sqrt{\frac{\mathrm{CS}(W, Y_r)\,\mathrm{CS}(W, Y_s)}{\mathrm{CS}(Y_r, Y_s)}}. \tag{8.5}$$

*Moreover, $Z_{rs}$ is the unique ray $Z$ in $[Y_r, Y_s]$ for which $\mathrm{CS}(W,Z) = \mu_W(C)$.*

*Proof.* By the arguments in the proof of Theorem 8.1, for Case B, we have

$$\mathrm{CS}(W, Z_{rs}) \leqslant \mathrm{CS}(W, Y) = \mu_W(C).$$

Trivially

$$\mu_W(C) \leqslant \mu_W(Y_r, Y_s) \leqslant \mathrm{CS}(W, Z_{rs}).$$

We conclude that equality holds here everywhere. Since $\mu_W(Y_r, Y_s) = \mu_W(C)$ is smaller than both $\mathrm{CS}(W, Y_r)$ and $\mathrm{CS}(W, Y_s)$, the function $\mathrm{CS}(W, -)$ on $[Y_r, Y_s]$ is certainly not monotone. We conclude by Theorem 7.7.d, that the most right equality in (8.5) holds, as well as the last assertion in the theorem. $\square$

**Corollary 8.3.** *Assume that $Y_1, \dots, Y_n, W, W'$ are rays in $V$ with $\mathrm{CS}(W, Y_i) = \mathrm{CS}(W', Y_i)$ for $1 \leqslant i \leqslant n$. Then*

$$\mu_W(Y_1, \dots, Y_n) = \mu_{W'}(Y_1, \dots, Y_n).$$

*Proof.* Let $C = \mathrm{conv}(Y_1, \dots, Y_n)$. We shall infer from §4, §5, and Theorem 8.1 that the value $\mu_W(C) = \mu_W(Y_1, \dots, Y_n)$ is uniquely determined by the quantities $\mathrm{CS}(W, Y_i)$, $1 \leqslant i \leqslant n$. This is trivial for $n = 1$, while for $n = 2$,

$$\mu_W(C) = \min(\mathrm{CS}(W, Y_1), \mathrm{CS}(W, Y_2))$$

except in the case that the profile of $\mathrm{CS}(W, -)$ on $[Y_1, Y_2]$ is not monotone. This property only depends on the values $\mathrm{CS}(W, Y_1)$ and $\mathrm{CS}(W, Y_2)$ (cf. Table 4.3). Then

$$\mu_W(C) = \sqrt{\frac{\mathrm{CS}(W, Y_1)\,\mathrm{CS}(W, Y_2)}{\mathrm{CS}(Y_1, Y_2)}}.$$

When $n \geqslant 3$ it follows from Theorem 8.1 (and more explicitly from Theorem 8.2) that $\mu_W(C)$ is determined by the values $\mathrm{CS}(W, Y_i)$, $1 \leqslant i \leqslant n$, and those values $\mu_W(Y_r, Y_s)$, $1 \leqslant r < s \leqslant n$, which are smaller than $\mathrm{CS}(W, Y_r)$ and $\mathrm{CS}(W, Y_s)$. Thus in all cases $\mu_W(C)$ remains unchanged if we replace $W$ by $W'$. $\square$

Formula (8.3) has been the main new ingredient for proving Theorems 8.1 and 8.2. We quote another (immediate) consequence of this formula.

**Theorem 8.4.** *Assume the $eR$-module $eV$ is free with base $y_1, \dots, y_n$. Let $y \in eV$, $Y = \operatorname{ray}(y)$, and $Y_i = \operatorname{ray}(y_i)$. Then*

$$\mathrm{CS}(W, Y) \leqslant \frac{q_{\mathrm{QL}}(y)}{q(y)} \cdot \max_{1 \leqslant i \leqslant n} \mathrm{CS}(W, Y_i). \tag{8.6}$$

This theorem is a sharpening of [7, Theorem 7.9.a] in the free case. Conversely it is immediate to deduce the quoted result in [7] from (8.6) by pulling back the quadratic pair $(eq, eb)$ to a free module.

We now study the minimal values of $\mathrm{CS}(W, -)$ on the convex hulls of subsets of $\{Y_1, \dots, Y_n\}$.

**Definition 8.5.** *Given a finite set $S = \{Y_1, \dots, Y_n\}$ of rays in $V$ and a ray $W$ in $V$, we define the subset*

$$\overrightarrow{\mu}_W(S) := \overrightarrow{\mu}_W(Y_1, \dots, Y_n),$$

*of $eR$ as follows:*

$$\overrightarrow{\mu}_W(S) := \{\mu_W(T) \mid T \subset S, T \neq \varnothing\}, \tag{8.7}$$

*i.e., $\overrightarrow{\mu}_W(S)$ is the set of minimal values of $\mathrm{CS}(W, -)$ on the convex hulls of all nonempty subsets of $S$. We call $\overrightarrow{\mu}_W(S)$ the CS-**spectrum** of $W$ on the set of rays $S$.*

We list the finite poset $\overrightarrow{\mu}_W(S)$ as a sequence

$$\mu_W^0(S) < \mu_W^1(S) < \cdots < \mu_W^m(S) \tag{8.8}$$

in $eR$. Here $\mu_W^0(S)$ and $\mu_W^m(S)$ are the minimum and the maximum, respectively, of $\mathrm{CS}(W, -)$ on the convex hull $C = \operatorname{conv}(S)$ of $S$. Notice that the other values $\mu_W^i(S)$ will often depend on the set of generators $S$ of the convex set $C$ instead of $C$ alone.

Aa a consequence of Theorem 8.1 we have the following fact.

**Scholium 8.6.** *Let $S = \{Y_1, \dots, Y_n\}$. The elements of $\overrightarrow{\mu}_W(S)$ are the values $\mathrm{CS}(W, Y_i)$, $1 \leqslant i \leqslant r$, and $\mathrm{CS}(W, M_W(Y_r, Y_s))$ where $(Y_r, Y_s)$ runs through all pairs in $S$ such that $\mathrm{CS}(W, -)$ is not monotone on $[Y_r, Y_s]$.*

**Proposition 8.7.** *Assume that $P$ and $Q$ are subsets of $S$, such that all intervals $[Y, Z]$ with $Y \in P$, $Z \in Q$ have a monotone $W$-profile. Then*

$$\overrightarrow{\mu}_W(P \cup Q) = \overrightarrow{\mu}_W(P) \cup \overrightarrow{\mu}_W(Q). \tag{8.9}$$

*In particular, this holds **for every** $W \in \operatorname{Ray}(V)$, if the quadratic form $eq$ is quasilinear on these intervals $[Y, Z]$.*

*Proof.* This is evident from the preceding description of CS-spectra. □

## 9. The glens and the glen locus of a finite set of rays

As previously, we assume that the ghost ideal $eR$ of the supertropical semiring $R$ is a semifield and that the quadratic form $q$ on $V$ is anisotropic.

**Definition 9.1.** *The **glen** of a finite sequence of rays $Y_1, \dots, Y_n$ in $V$ **at a ray** $W$ in $V$ is the set of all $Z \in \operatorname{conv}(Y_1, \dots, Y_n)$ such that*

$$\mathrm{CS}(W, Z) < \min_{1 \leqslant i \leqslant n} \mathrm{CS}(W, Y_i).$$

*We denote this set by $\mathrm{Glen_W(Y_1, \dots, Y_n)}$, and call it the $W$-**glen of** $(Y_1, \dots, Y_n)$, for short.*

For notational reasons we do not exclude the case $n = 1$. Then, of course, all $W$-glens are empty.

**Proposition 9.2.** $\mathrm{Glen_W(Y_1, \dots, Y_n)}$ *is a convex subset of* $\mathrm{Ray}(V)$ *(perhaps empty).*

*Proof.* Given three rays $Z_1, Z_2, Z$ in $\mathrm{conv}(Y_1, \dots, Y_n)$ with $Z \in [Z_1, Z_2]$ and $\mathrm{CS}(W, Z_1) < \mathrm{CS}(W, Y_i)$, $\mathrm{CS}(W, Z_2) < \mathrm{CS}(W, Y_i)$ for all $i \in [1, n]$, we infer from Theorem 8.4 that $\mathrm{CS}(W, Z) < \mathrm{CS}(W, Y_i)$ for all $i \in [1, n]$. □

For $n = 2$ and $Y_1 \neq Y_2$ the rays $Y_1$ and $Y_2$ are uniquely determined, up to permutation, by the closed interval $[Y_1, Y_2]$, as we know. Thus we are entitled to define the $W$**-glen of** $[Y_1, Y_2]$ as

$$\mathrm{Glen_W[Y_1, Y_2]; = Glen_W(Y_1, Y_2)}.$$

From the analysis of the function $CS(W, -)$ on closed intervals in §3 we infer the following statement, which justifies the use of the name "glen" at least for $n = 2$. See also Figure 4 below.

**Scholium 9.3.** $\mathrm{Glen_W[Y_1, Y_2]}$ *is not empty iff the $W$-profile on $[Y_1, Y_2]$ is not monotonic, and thus is of type $B$ or $B'$ or $\partial B$ (cf. §4).*

Relying on §3, we give an explicit description of the $W$-glen of $[Y_1, Y_2]$ in the case of type $B$ or $\partial B$, i.e., when $\mathrm{CS}(W, -)$ is not monotonic on $[Y_1, Y_2]$ and $\mathrm{CS}(W, Y_1) \leqslant (W, Y_2)$. Choosing vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in V$ such that $W = \mathrm{ray}(\varepsilon_1)$, $Y_1 = \mathrm{ray}(\varepsilon_2)$, $Y_2 = \mathrm{ray}(\varepsilon_3)$, we have the following illustration of the function $f(\lambda) = \mathrm{CS}(\varepsilon_1, \varepsilon_2 + \lambda\varepsilon_3) = f_1(\lambda) + f_2(\lambda)$, using the notations from §3.

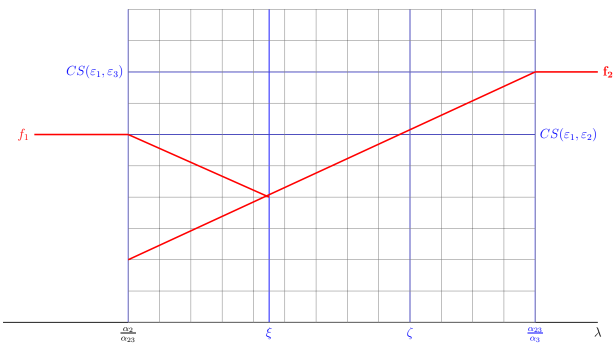


FIGURE 4. $\xi = \frac{\alpha_{12}}{\alpha_{13}}$.

The $W$-glen of $[Y_1, Y_2]$ is contained in the open interval $]Y_{12}, Y_{21}[$ , where $Y_{12}$ is the characteristic ray of $[Y_1, Y_2]$ near $Y_1$ and $Y_{21}$ is the characteristic ray near $Y_2$. It starts at the argument $\lambda = \frac{\alpha_2}{\alpha_{23}}$ corresponding to $Y_{12}$ and ends at the argument $\zeta \in ]\frac{\alpha_2}{\alpha_{23}}, \frac{\alpha_{23}}{\alpha_3}[$ with $f_2(\zeta) = \mathrm{CS}(\varepsilon_1, \varepsilon_2)$. In the interval $[\frac{\alpha_2}{\alpha_{23}}, \frac{\alpha_{23}}{\alpha_3}]$ the function $f_2$ reads

$$f_2(\lambda) = \frac{\lambda^2\alpha_{13}^2}{\alpha_1 \cdot \lambda\alpha_{23}} = \frac{\lambda\alpha_{13}^2}{\alpha_1 \cdot \alpha_{23}}$$

(cf. (3.2)), since here the term $\lambda\alpha_{23}$ is dominant in the formula (3.7) for $q(\varepsilon_2 + \lambda\varepsilon_3)$. Thus we have to solve

$$\frac{\zeta\alpha_{13}^2}{\alpha_1\alpha_{23}} = \frac{\alpha_{12}^2}{\alpha_1\alpha_2},$$

and obtain
$$\zeta = \frac{\alpha_{12}^2}{\alpha_{13}^2} \cdot \frac{\alpha_{23}}{\alpha_2}. \tag{9.1}$$
In the subcase $\mathrm{CS}(\varepsilon_1, \varepsilon_2) = \mathrm{CS}(\varepsilon_1, \varepsilon_2)$, i.e., $\frac{\alpha_{12}^2}{\alpha_1\alpha_2} = \frac{\alpha_{13}^2}{\alpha_1\alpha_3}$, we obtain
$$\zeta = \frac{\alpha_{23}}{\alpha_3}. \tag{9.2}$$

Introducing the ray
$$Z_{21} := \mathrm{ray}(\varepsilon_2 + \zeta\varepsilon_3) = \mathrm{ray}(\alpha_{13}^2\alpha_2\varepsilon_2 + \alpha_{12}^2\alpha_{23}\varepsilon_3),$$
we summarize our study of the $W$-glen of $[Y_1, Y_2]$ as follows.

**Theorem 9.4.** *Assume that* $\mathrm{CS}(W, Y_1) \leqslant \mathrm{CS}(W, Y_2)$ *and* $\mathrm{Glen_W[Y_1, Y_2]} \neq \emptyset$. *Then*

(a) $\mathrm{Glen_W[Y_1, Y_2] = ]Y_{12}, Z_{21}[ \subset ]Y_{12}, Y_{21}[}$,

(b) $\mathrm{Glen_W[Y_1, Y_2] = ]Y_{12}, Y_{21}[}$ *iff* $\mathrm{CS}(W, Y_1) = \mathrm{CS}(W, Y_2)$.

**Remark 9.5.** *Note that*
$$\sqrt{\frac{\alpha_2}{\alpha_{23}} \cdot \frac{\alpha_{12}^2}{\alpha_{13}^2} \cdot \frac{\alpha_{23}}{\alpha_2}} = \frac{\alpha_{12}}{\alpha_{13}} = \xi.$$
*Thus the median $M_W(Y_1, Y_2)$ may be viewed as a kind of geometric mean of the rays $Y_{12}$ and $Z_{21}$.*

We now look at the set of rays $W$ where a nonempty glen of $(Y_1, \dots, Y_n)$ occurs.

**Definition 9.6.** *The* ***glen-locus*** *of* $(Y_1, \dots, Y_n)$ *is the set*
$$\begin{aligned} \mathrm{Loc_{glen}(Y_1, \dots, Y_n)} &:= \{W \in \mathrm{Ray}(V) \mid \mathrm{Glen_W(Y_1, \dots, Y_n)} \neq \emptyset\} \\ &= \big\{W \in \mathrm{Ray}(V) \mid \mu_W(Y_1, \dots, Y_n) < \min_{1 \leqslant i \leqslant n} \mathrm{CS}(W, Y_i)\big\}. \end{aligned}$$

For $n = 2$, $Y_1 \neq Y_2$, we define
$$\mathrm{Loc_{glen}[Y_1, Y_2] := Loc_{glen}(Y_1, Y_2)},$$
which makes sense, since the rays $Y_1, Y_2$ are uniquely determined, up to permutation, by the interval $[Y_1, Y_2]$. Theorem 8.2 translates into the following statement, where we use the definition of loci of basic and composed profile types from §5 (Definitions 5.5 and 5.7).

**Scholium 9.7.** $\mathrm{Loc_{Glen}[Y_1, Y_2]}$ *is the disjoint union of the basic loci* $\mathrm{Loc_B}[\overrightarrow{Y_1, Y_2}]$, $\mathrm{Loc_{B'}}[\overrightarrow{Y_1, Y_2}]$, *and* $\mathrm{Loc_{\partial B}[Y_1, Y_2]}$, *which are disjoint convex subsets of* $\mathrm{Ray}(V)$.*It is also the union of the composed loci* $\mathrm{Loc_{\overline{B}}}[\overrightarrow{Y_1, Y_2}]$, $\mathrm{Loc_{\overline{B'}}}[\overrightarrow{Y_1, Y_2}]$, *which are again convex, and have the intersection* $\mathrm{Loc_{\partial B}[Y_1, Y_2]}$ *(cf. Theorems 5.6 and 5.8).*

Of course it may happen that $\mathrm{Loc_{Glen}[Y_1, Y_2]}$ is empty. In particular this occurs if $[Y_1, Y_2]$ is $\nu$-quasilinear.

Given a set $\{Y_1, \dots, Y_n\}$ of rays in $V$ with $n > 2$, we have the important fact in consequence of Theorem 8.1 (cf. Scholium 8.6), that $\mathrm{Loc_{glen}(Y_1, \dots, Y_n)}$ is contained in the union of all sets $\mathrm{Loc_{glen}(Y_r, Y_s)}$ with $1 \leqslant r < s \leqslant n$, such that $[Y_r, Y_s]$ is $\nu$-excessive [7, Definition 7.3], equivalently, $\mathrm{Loc_{glen}[Y_r, Y_s]} \neq \emptyset$. If there are $u = u(Y_1, \dots, Y_n)$ such pairs $(r, s)$, then $\mathrm{Loc_{glen}(Y_1, \dots, Y_n)}$ is a union of at most $2u$ convex subsets of $\mathrm{Ray}(V)$.

## 10. Explicit description of the set of minima of a CS-function on a finitely generated convex set in the ray space

*In the whole section $R$ is a supertropical semiring, $eR$ is a nontrivial semifield, and $(q,b)$ is a quadratic pair on an $R$-module $V$ with $q$ anisotropic.* Given a finite subset $S \subset \mathrm{Ray}(V)$ and a fixed ray $W$ in $V$, we explore the set of minima of $\mathrm{CS}(W,-)$ on the convex hull $C$ of $S$ in $\mathrm{Ray}(V)$, denoted by $\mathrm{Min}\,\mathrm{CS}(W,C)$. We already know that $\mathrm{Min}\,\mathrm{CS}(W,C)$ is nonempty. Our first goal is to prove that $\mathrm{Min}\,\mathrm{CS}(W,C)$ is again a finitely generated convex subset of $\mathrm{Ray}(V)$, and to determine a set of generators of $\mathrm{Min}\,\mathrm{CS}(W,C)$ starting from $S$.

**Theorem 10.1.**

(a) $\mathrm{Min}\,\mathrm{CS}(W,C)$ *is convex.*

(b) *If* $X \in \mathrm{Min}\,\mathrm{CS}(W,C)$ *and* $Y \in C \setminus \mathrm{Min}\,\mathrm{CS}(W,C)$, *then* $[X,Y] \cap \mathrm{Min}\,\mathrm{CS}(W,C) = [X, M_W(X,Y)]$.

(c) *If* $\mathrm{CS}(W,-)$ *is constant on* $S$, *then* $\mathrm{CS}(W,-)$ *is constant on* $C$.

(d) *Assuming that* $\mathrm{CS}(W,-)$ *is not constant on* $S$. *Let* $S^*$ *denote the set of all medians* $M_W(X,Y)$ *with* $X,Y \in S$ *and* $\mathrm{CS}(W, M_W(X,Y)) = \mu_W(S)$, [14] *which may be empty. Let*
$$P := (S \cup S^*) \cap \mathrm{Min}\,\mathrm{CS}(W,C),$$
$$Q := S \setminus P = (S \cup S^*) \setminus P.$$
*Then* $\mathrm{Min}\,\mathrm{CS}(W,C)$ *is the convex hull of the finite set* $P \cup M_W(P,Q)$, *where*
$$M_W(P,Q) := \{M_W(X,Y) \mid X \in P, Y \in Q\}.$$

*Proof.* (a): If $X,Y$ are rays in $\mathrm{Min}\,\mathrm{CS}(W,C)$, then $\mathrm{CS}(W,X) = \mathrm{CS}(W,Y)$, from which we conclude that $\mathrm{CS}(W,-)$ is constant on $[X,Y]$, since no $W$-glen is possible on $[X,Y]$. Thus $[X,Y] \subset \mathrm{Min}\,\mathrm{CS}(W,C)$.

(b): $\mathrm{CS}(W,Y)$ attains its minimal value on $[X,Y]$ in $X$. Thus it is clear from our analysis of $\mathrm{CS}(W,-)$ on $[\overrightarrow{X,Y}]$ in §3, that the set $Z$ of rays in $[X,Y]$ with $\mathrm{CS}(W,Z) = \mathrm{CS}(W,X)$ is $[X, M_W(X,Y)]$, cf. (3.11) and (3.12).

(c): Assume that $\mathrm{CS}(W,-)$ is constant on $S$, then $\mathrm{CS}(W,-)$ has no glens at all, and we conclude as in (a) that $\mathrm{CS}(W,-)$ is constant on $C$.

(d): It follows from (b) that $M_W(P,Q)$ is contained in $\mathrm{Min}\,\mathrm{CS}(W,C)$ and then by (a) that
$$\mathrm{conv}(P \cup M_W(P,Q)) \subset \mathrm{Min}\,\mathrm{CS}(W,C).$$
We now verify that any given ray $Y \in \mathrm{Min}\,\mathrm{CS}(W,C)$ is contained in the convex hull of $P \cup M_W(P,Q)$. Let $S = \{Y_1, \dots, Y_n\}$. We choose a minimal subset $\{Y_k \mid k \in K\}$ of $S$, $K \subset \{1, \dots, n\}$, such that
$$Y \in \mathrm{conv}(P \cup \{Y_k \mid k \in K\}). \tag{10.1}$$
If $K = \emptyset$, then $Y \in \mathrm{conv}(P)$, and we are done.

In the case that $K \neq \emptyset$ we choose a minimal set of rays $\{Z_j \mid j \in J\}$ in $P$ such that
$$Y = \mathrm{conv}(\{Z_j \mid j \in J\} \cup \{Y_k \mid k \in K\}), \tag{10.2}$$

[14] Recall that $\mu_W(S) = \mu_W(C)$ denotes the minimal value of $\mathrm{CS}(W,-)$ on $S$, and hence on $C$.

so that $Y \in \operatorname{conv}(A \cup B)$ where

$$A = \operatorname{conv}(\{Z_j \mid j \in J\}), \qquad B = \operatorname{conv}(\{Y_k \mid k \in K\}).$$

Then $A \subset P \subset \operatorname{Min}\operatorname{CS}(W,C)$ while $B$ is disjoint from $P$ due to the minimality of the set $\{Y_k \mid k \in K\}$ in (10.1). Since $Y \in \operatorname{Min}\operatorname{CS}(W,C)$, we conclude by assertion (b) that $Y \in M_W(Z,T)$ for some rays $Z \in A$, $T \in B$. Choosing vectors $z_j \in eZ_j$, $y_k \in eY_k$, $w \in eW$ we have $Y = \operatorname{ray}(y)$, with

$$y = m_w\Big(\sum_{j\in J}\mu_j z_j, \sum_{k\in K}\lambda_k y_k\Big) \tag{10.3}$$

and nonzero coefficients $\mu_j, \lambda_k \in eR$. Thus (cf. (7.1) and (7.2))

$$\begin{aligned} y &= b\Big(w, \sum_{k\in K}\lambda_k y_k\Big)\sum_{j\in J}\mu_j z_j + b\Big(w, \sum_{j\in J}\mu_j z_j\Big)\sum_{k\in K}\lambda_k y_k \\ &= \sum_{j\in J, k\in K}\mu_j\lambda_k\big(b(w,y_k)z_j + b(w,z_j)y_k\big), \end{aligned}$$

which proves that

$$Y \in \operatorname{conv}\big(\{M_W(Z_j,Y_k) \mid j \in J, k \in K\}\big) \subset \operatorname{conv}(M_W(P,Q)). \tag{10.4}$$

□

It is to be expected from Theorem 10.1 that usually many more rays are needed to generate the convex set $\operatorname{Min}\operatorname{CS}(W,C)$ than to generate $C$, But now we exhibit cases, where $\operatorname{Min}\operatorname{CS}(W,C) = \operatorname{Min}\operatorname{CS}(W,S)$ can be generated by very few rays.

**Proposition 10.2.** *If $\mu_W(S) = 0$, then $\operatorname{Min}\operatorname{CS}(W,C)$ is the convex hull of*

$$W^\perp \cap S = \{Z \in S \mid \operatorname{CS}(W,Z) = 0\}.$$

*Proof.* By the results in §7 there are no rays $X, Y$ in $V$ with $\operatorname{CS}(W,X) > 0$, $\operatorname{CS}(W,Y) > 0$, $\operatorname{CS}(W, M_W(X,Y)) = 0$ (cf. e.g. (7.10)). Thus $S^* = 0$, and we conclude from Theorem (10.1) that $\operatorname{Min}\operatorname{CS}(W,C) = \operatorname{conv}(W^\perp \cap S)$. □

**Example 10.3.** *Assume that $S = \{X_1, \dots, X_n\}$, $n \geqslant 2$, is a finite set of rays in $V$ such that there exists a ray $Z$ with $0 < \operatorname{CS}(W,Z) < \operatorname{CS}(W,X_i)$ for every $i \in \{1,\dots,n\}$ and $M_W(X_i,X_j) = Z$ for $1 \leqslant i < j \leqslant n$. Then $S^* = \{Z\}$, and $M_W(X_i,X_j) = Z$ implies that $\operatorname{CS}(W,-)$ is strictly increasing on $[\overrightarrow{Z,X_i}]$ (and $[\overrightarrow{Z,X_j}]$), cf. §3, whence $M_W(Z,X_i) = Z$ for every $i$, Thus $\operatorname{CS}(W,S) = \{Z\}$.*

**Definition 10.4.** *We call a set $S$ as described in Example 10.3 a* ***median cluster for*** *$W$, or $W$-median cluster,* ***with apex*** *$Z$.*

**Example 10.5.** *Assume that $S = P_1 \cup P_2$ is a disjoint union of two $W$-median clusters $P_1$ and $P_2$ with apices $Z_1$ and $Z_2$. Assume further that all pairs $X, Y$ with $X \in P_1$, $Y \in P_2$ have the same median $M_W(X,Y) = Z_{12}$ and that*

$$\operatorname{CS}(W,Z_1) = \operatorname{CS}(W,Z_2) = \operatorname{CS}(W,Z_{12}).$$

*Then we conclude from Theorem 10.1 that*

$$\operatorname{Min}\operatorname{CS}(W,S) = \operatorname{conv}(Z_1, Z_2, Z_{12}). \tag{10.5}$$

*Indeed, in the notation there* $P = S^* = \{Z_1, Z_2\}$, $Q = S = P_1 \cup P_2$, *and so* $M_W(P, Q) = \{Z_1, Z_2, Z_{12}\}$. *In the case* $Z_1 = Z_2$ *we obtain*

$$\operatorname{Min} \operatorname{CS}(W, S) = [Z, Z_{12}] \tag{10.6}$$

*with* $Z := Z_1 = Z_2$.

Given a ray $Z$ in $V$, we now focus on the set of all $W$-median clusters in $\operatorname{Ray}(V)$ with apex $Z$. We assume that $\operatorname{CS}(W, Z) > 0$, since otherwise it is clear from Proposition 10.2, that there are no median clusters with apex $Z$. The next lemma, a simplification of an argument in the proof of Theorem 10.1.d ((10.1)–(10.4)), will be of help.

**Lemma 10.6** ($M_W$-Convexity Lemma)**.** *Let* $Z \in \operatorname{Ray}(V)$. [15] *Assume that* $P = \{Y_j \mid j \in K\}$ *and* $Q = \{Y_k \mid k \in J\}$ *are disjoint sets of rays with* $M_W(Y_j, Y_k) = Z$ *for any* $Y_j \in P$ *and* $Y_k \in Q$. *Then also* $M_W(Y, T) = Z$ *for any* $Y \in \operatorname{conv}(P)$ *and* $T \in \operatorname{conv}(Q)$.

*Proof.* Given $Y \in P$, $T \in Q$ we choose vectors $y_j \in eY_j$, $y_k \in eY_k$, $z \in eZ$, $t \in eT$. Then $Y = \operatorname{ray}(y)$, $T = \operatorname{ray}(t)$ with $y = \sum_{j \in J} \lambda_j y_j$, not all $\lambda_j = 0$, and $y = \sum_{k \in K} \mu_k y_k$, not all $\mu_k = 0$. Since $M_W(Y_j, Y_k) = Z$ for $j \in J$, $k \in K$, we have

$$b(w, y_k) y_j + b(w, y_j) y_k = m_w(y_j, y_k) = \alpha_{jk} z,$$

for these indices $j, k$, with $\alpha_{jk} \neq 0$. Thus

$$\begin{aligned} b(w,t)y + b(w,y)t &= \sum_{k \in K} \mu_k b(w, y_k) \sum_{j \in J} \lambda_j y_j + \sum_{j \in J} \lambda_j b(w, y_j) \sum_{k \in K} \mu_k y_k \\ &= \sum_{j \in J, k \in K} \lambda_j \mu_k [b(w, y_k) y_j + b(w, y_j) y_k] \\ &= \Big( \sum_{j \in J, k \in K} \alpha_{jk} \lambda_j \mu_k \Big) z. \end{aligned}$$

Since $\sum_{j \in J, k \in K} \alpha_{jk} \lambda_j \mu_k \neq 0$, this proves that $M_W(Y, T) = Z$. $\square$

Given a ray $Z$ in $V$ with $\operatorname{CS}(W, Z) > 0$, we introduce the ray set

$$Z^{\uparrow} := \{X \in \operatorname{Ray}(V) \mid \operatorname{CS}(W, X) > \operatorname{CS}(W, Z)\}. \tag{10.7}$$

This set contains every $W$-median cluster having apex $Z$. Note that typically the set $Z^{\uparrow}$ is not convex.

**Definition 10.7.** *Let* $P \subset Z$, $P \neq \emptyset$. *The* $Z$***-polar of*** $P$ ***for*** $W$ *(or* $W$*-*$Z$*-polar of* $P$*) is the set*

$$\check{P} = P^{\vee} := \{Y \in Z^{\uparrow} \mid \exists X \in P : M_W(X, Y) = Z\}. \tag{10.8}$$

Note that

$$P_1 \subset P_2 \subset Z^{\uparrow} \quad \Rightarrow \quad \check{P}_2 \subset \check{P}_1, \tag{10.9}$$

and, that

$$\Big( \bigcup_{\lambda \in \Lambda} P_\lambda \Big)^{\vee} = \bigcap_{\lambda \in \Lambda} \check{P}_\lambda \tag{10.10}$$

for any family $(P_\lambda \mid \lambda \in \Lambda)$ of subsets $P_\lambda$ of $Z^{\uparrow}$.

[15] Here it is not necessary to assume that $\operatorname{CS}(Z, W) > 0$.

**Remarks 10.8.** *Let $P \subset Z^{\uparrow}$, $P \neq \emptyset$.*

(a) *Then $P$ and $\check{P}$ are disjoint, since $M_W(X, X) = X \neq Z$ for every $X \in P$.*

(b) *If $\check{P} \neq \emptyset$, then $P \subset P^{\vee\vee}$, This implies in the usual way that*

$$P^{\vee\vee\vee} = P^{\vee}. \tag{10.11}$$

We define for $P \subset Z^{\uparrow}$ the set

$$\mathrm{conv}_0(P) := \mathrm{conv}(P) \cap Z^{\uparrow}. \tag{10.12}$$

**Theorem 10.9.** *Let $P \subset Z^{\uparrow}$ and $P \neq \emptyset$, then*

$$\check{P} = \mathrm{conv}_0(\check{P}) = \mathrm{conv}_0(P)^{\vee}.$$

*Proof.* We have $P \cap \check{P} = \emptyset$ and $M_W(X, Y) = Z$ for $X \in P$, $Y \in \check{P}$. By the Median Convexity Lemma 10.6, this implies $M_W(X', Y') = Z$ for $X' \in \mathrm{conv}(P)$ and $Y' \in \mathrm{conv}(\check{P})$. Thus $\mathrm{conv}_0(\check{P}) \subset \mathrm{conv}_0(P)^{\vee}$. We further infer from $P \subset \mathrm{conv}_0(P)$ that $\mathrm{conv}_0(P)^{\vee} \subset \check{P}$, and so $\mathrm{conv}_0(\check{P}) \subset \check{P}$. Since trivially $\check{P} \subset \mathrm{conv}_0(\check{P})$, this proves that $\check{P} = \mathrm{conv}_0(\check{P}) = \mathrm{conv}_0(P)^{\vee}$. □

We now employ the partial ordering $\leqslant_Z$ on $\mathrm{Ray}(V)$, given by

$$Y' \leqslant_Z Y \quad \Leftrightarrow \quad [Z, Y'] \subset [Z, Y],$$

the basics of which can be found in [8, §8]. This ordering extends the total ordering on the oriented intervals $[\overrightarrow{Z, Y}]$ used in the previous sections.

**Theorem 10.10.** *For any nonempty subset $P$ of $Z^{\uparrow}$ the $Z$-polar $\check{P}$ is compatible with $\leqslant_Z$ in the following sense. If $Y, Y' \in Z^{\uparrow}$ and $Y \leqslant_Z Y'$, then*

$$Y \in \check{P} \quad \Leftrightarrow \quad Y' \in \check{P}. \tag{10.13}$$

*Proof.* This follows from the fact that for any $X \in P$ the CS-function $\mathrm{CS}(W, -)$ is not monotonic on $[X, Y]$ iff it is not monotonic on $[X, Y']$, and then $\mathrm{CS}(W, -)$ attains its unique minimum at $M_W(X, Y) = M_W(X, Y')$, as is clear from §3 and §4, cf. Figures 1–3 in §3. □

We describe a procedure to build up clusters with apex $Z$, basing on some more terminology. For any ray $X \in Z^{\uparrow}$, we write $\check{X} = X^{\vee} = \{X\}^{\vee}$ for short.

**Definition 10.11.** *We say that $X$ is $Z$-polar, if $\check{X} \neq \emptyset$, and so $X$ is in the $Z$-polar of the set $\check{X}$. More explicitly, $X$ is $Z$-polar, if $M_W(X, Y) = Z$ for some $Y \in Z^{\uparrow}$.*

Note that for any set $P \subset Z^{\uparrow}$ we have

$$\check{P} = \bigcap_{X \in P} \check{X}. \tag{10.14}$$

If $Z^{\uparrow}$ does not contain $Z$-polar sets, then, of course, there do not exist median clusters with apex $Z$. Otherwise we choose $X_1, X_2 \in Y^{\uparrow}$ with $M_W(X_1, X_2) = Z$. If $\{X_1, X_2\}^{\vee} = X_1^{\vee} \cap X_2^{\vee} \neq \emptyset$, we choose a ray $X_3 \in Z^{\uparrow}$ with $X_3 \in \{X_1, X_2\}^{\vee}$. Proceeding in this way we obtain a sequence of rays $X_1, \dots, X_r$ in $Y^{\uparrow}$ with $r \geqslant 2$ and

$$X_{i+1} \in \{X_1, \dots, X_i\}^{\vee} \qquad \text{for } 1 \leqslant i < r. \tag{10.15}$$

There are two cases.

Case A: We reach a set $S = \{X_1, \dots, X_r\}$ with $\{X_1, \dots, X_r\}^\vee = \check{X}_1 \cap \dots \cap \check{X}_r = \emptyset$. Then $S$ is a maximal median cluster with apex $Z$.

Case B: We obtain infinite sets $S \subset Z^\uparrow$, such that every finite subset $T \subset S$, $|T| \geqslant 2$, is a $W$-median cluster with apex $Z$. We call such set $S$ a **generalized $W$-median cluster** with apex $Z$ (or generalized $W$-$Z$-median cluster). More specifically, using mild set theory, we obtain by a transfinite induction procedure a sequence of rays $\{X_i \mid 1 \leqslant i \leqslant \lambda\}$ with ordinal $\lambda \geqslant \omega$ which is a **maximal generalized $W$-$Z$-median cluster**.

## 11. The equal polar relation

Let $W$ and $Z$ be any rays in $V$. Given $X_1, X_2 \in Z^\uparrow$, cf. (10.7), we say that $X_1$ and $X_2$ are **$W$-$Z$-equivalent** (or $Z$-equivalent for short), and write $X_1 \sim_Z X_2$, if $\check{X}_1 = \check{X}_2$. We call this equivalence relation on $Z^\uparrow$ the **equal polar relation** for $W$ and $Z$ (or the **$W$-$Z$-equivalence relation**). For this relation, the equivalence class of a ray $X \in Z^\uparrow$ is denoted by

$$[X] := [X]_Z := [X]_{W,Z}.$$

Note that, if $\check{X}_1 \neq \emptyset$, then $X_1 \sim_Z X_2$ iff $X_1^{\vee\vee} = X_2^{\vee\vee}$, cf. (10.11). We then abbreviate $X^{\vee\vee} = \widetilde{X}$.

For most problems concerning $Z$-polars of rays, and in particular all problems appearing in §10, only the class $[X]_{W,Z}$ matters. For example, in a (generalized, maximal) median cluster $P$ with apex $Z$ we may replace any $X \in P$ by an $Z$-equivalent ray $X'$, and have again a (generalized, maximal) median cluster $P'$ with apex $Z$. Therefore, understanding the $W$-$Z$-equivalence is a very basic goal, which we first pose vaguely as follows.

**Problem 11.1.** *Describe the pattern of any $Z$-equivalence classes $[X] \subset Z^\uparrow$.*

To approach this problem, so far, we only know:

(a) All rays $X$ with $\check{X} = \emptyset$ are in one equivalence class – the class of non-polar rays (Definition 10.11). This is trivial. We denote this class by $C_\emptyset$:

$$C_\emptyset = \{X \in Z^\uparrow \mid \forall Y \in Z^\uparrow : M_W(X, Y) \neq Z\}.$$

Perhaps it is best to discard $C_\emptyset$ from $Z^\uparrow$.

(b) $[X]_Z \subset \widetilde{X}$. Indeed, if $X_1^\vee = X_2^\vee$, then $X_1 \in X_1^{\vee\vee} = X_2^{\vee\vee}$.

(c) The relation $\sim_Z$ is compatible with the partial ordering $\leqslant_Z$ on $Z^\uparrow$, i.e., if $X_1$ and $X_2$ are comparable under $\leqslant_Z$, then $X_1 \sim_Z X_2$, cf. Theorem 10.10.

We now can point more precisely at the type of questions arising from Problem 11.1. If $\check{X} \neq \emptyset$, then $\widetilde{X}$ is a convex subset of $\mathrm{Ray}(V)$ contained in $Z^\uparrow$, with $[X] \subset \widetilde{X}$ by (b). If $T \in \widetilde{X}$, then $[T] \subset \widetilde{T} \subset \widetilde{X}$, and so $\widetilde{X}$ *is the disjoint union of all classes* $[T]$ *contained in* $\widetilde{X}$. Furthermore, since $\widetilde{T}$ is convex, also the convex hull $\mathrm{conv}([T])$ of the set $[T]$ is contained in $\widetilde{X}$. This leads to the next two intriguing questions.

A) Is $\mathrm{conv}([T])$ also a union of $Z$-equivalence classes?

B) When is a class $[T]$ by itself convex?

Due to (c) the whole pattern of classes $[T]$ is compatible with the partial ordering $\leqslant_Z$.

Concerning question B), so far we have only a partial answer.

**Theorem 11.2.** *If $X$ is a $Z$-polar ray in $Z^{\uparrow}$ (i.e., $\check{X} \neq \varnothing$), then (cf. (10.13))*

$$[X] = \operatorname{conv}_0\big([X]\big) = \operatorname{conv}\big([X]\big) \cap Z^{\uparrow}.$$

*Proof.* We need to prove the following. If $X_1, X_2 \in Z^{\uparrow}$, $[X_1, X_2] \in Z^{\uparrow}$, and $\check{X}_1 = \check{X}_2 \neq \varnothing$, then $\check{X}_1 = \check{T}$ for any $T \in [X_1, X_2]$. We have to verify for any $Y \in Z^{\uparrow}$ that

$$M_W(X_1, Y) = Z \quad \Leftrightarrow \quad M_W(T, Y) = Z.$$

($\Rightarrow$): If $M_W(X_1, Y) = Z$, then $M_W(X_2, Y) = Z$, since $\check{X}_1 = \check{X}_2$, whence by the $M_W$-Convexity Theorem: $M_W(T, Y) = Z$ for any $T \in [X_1, X_2]$.

($\Leftarrow$): Let $S = \{X_1, X_2, Z\}$. The CS-function $\mathrm{CS}(W, -)$ is strictly decreasing on $[\overrightarrow{X_1, Z}]$ and on $[\overrightarrow{X_2, Z}]$, furthermore $\mathrm{CS}(W, T) > \mathrm{CS}(W, Z)$ and $\mathrm{CS}(W, Y) > \mathrm{CS}(W, Z)$. We conclude from this that $\mathrm{CS}(W, -)$ is not monotonic on $[\overrightarrow{T, Y}]$ and has there minimum value $\mathrm{CS}(W, Z)$. This implies $M_W(T, Y) = Z$. □

We introduce two more notations around $Z$-equivalence. Recall that $X \sim_Z T \;\Leftrightarrow\; \widetilde{X} = \widetilde{T}$ (provided that $\check{X} \neq \varnothing$).

**Definition 11.3.** *A **path** in a class $[X]_Z$ is a sequence of rays $X_0, \dots, X_r$ in $[X]_Z$ where $[X_{i-1}, X_i] \in Z^{\uparrow}$ for $0 < i \leqslant r$, $r \geqslant 1$.*

Note that, in consequence of Theorem 11.2,

$$\bigcup_{i=1}^{r} [X_{i-1}, X_i] \subset [X]_Z. \tag{11.1}$$

This gives us an obvious notion of **path components** of $[X]_Z$. More generally, we may define paths and path components in any subset of $Z^{\uparrow}$.

**Definition 11.4.** *Given rays $X \in Z^{\uparrow}$ and $T \in [X]_Z$, we define the **median star** (=$W$-$Z$-median star) $\mathrm{st}_T(X)$ as the set of all rays $T'$ with $[T, T'] \subset [X]_Z$.*

In other words, $\mathrm{st}_T(X)$ is the union of all intervals $[T, T']$ contained in $[X]_Z$.

**Remark 11.5.** *If $T', T'' \in \mathrm{st}_T(X)$, then perhaps $[T', T''] \not\subset Z^{\uparrow}$. But, if $[T', T''] \subset Z^{\uparrow}$, then $\operatorname{conv}(T, T', T'') \subset [X]_Z$, and so $\operatorname{conv}(T, T', T'') \subset \mathrm{st}_T(X)$. Note also that*

$$\mathrm{st}_T([X]_Z) \subset \widetilde{T} \subset \widetilde{X}. \tag{11.2}$$

Every $Z$-equivalence class $[X]_Z$ is the disjoint union of the path components contained in $[X]_Z$. If $A$ and $B$ are such path components, then obviously every interval $[Y_1, Y_2]$ with $Y_1 \in A$, $Y_2 \in B$ has a "**deep glen**" with respect to $Z$, i.e., the median $M_W(Y_1, Y_2)$ is not contained in $Z^{\uparrow}$. We can refine Problem 11.1 to a description of the pattern of path components of the $W$-$Z$-equivalence classes, which we call the **refined version of Problem 11.1**. This seems to be natural and easier than Problem 11.1 above. Note also that every such path component is the union of all median stars $\mathrm{st}_T(X)$ contained in it.

We have gained a very rough view to the family of path components of $Z$-equivalence classes as follows. For simplicity, we assume that $eR = \{0\} \cup \mathcal{G}$ is a nontrivial bipotent semifield which is **square-root closed**, i.e., the injective endomorphism $x \mapsto x^2$ is also surjective, and so is an order preserving automorphism of $eR$. This setup can be reached for any (nontrivial) bipotent semifield by a canonical extension involving only square-roots, cf. [6, §7].

Assume that $A$ and $B$ are different sets, which are path components of $Z$-equivalence classes different from $C_\varnothing$. Then for any $T_1 \in A$ and $T_2 \in B$ the interval $[T_1, T_2]$ has a deep glen, and so we have a decomposition of $[T_1, T_2]$ into subintervals

$$[T_1, T_2] = [T_1, T_{12}[ \ \cup \ [T_{12}, T_{21}] \ \cup \ ]T_{21}, T_2] \tag{11.3}$$

such that

$$\begin{aligned} [T_1, T_2] \cap \mathrm{st}_{Y_1}(A) &= [T_1, T_{12}[ \ , \\ [T_1, T_2] \cap \mathrm{st}_{Y_2}(B) &= ]T_{21}, T_2], \end{aligned} \tag{11.4}$$

with

$$M_W(T_1T_2) = M_W[T_{12}, T_{21}] \notin Y^\uparrow, \tag{11.5}$$

$$\mathrm{CS}(W, T_{12}) = \mathrm{CS}(W, Z) = \mathrm{CS}(W, T_{21}). \tag{11.6}$$

This subdivision can be deduced from the defining formula (1.3) of a CS-ratio and the formulas for $M_W(T_1, T_2)$ in §3 in the case that $\mathrm{CS}(W, -)$ is not monotone on $[T_1, T_2]$, and the formulas of the glen of $[T_1, T_2]$ in §9. These formulas show that square roots suffice for the above subdivision. We omit the details.

To store the facts (11.3)–(11.6), we say, that the set of all path components of $Z$-equivalence classes $\neq C_\varnothing$ is the $W$-$Z$-**archipelago** in $\mathrm{Ray}(V)$ (for given rays $W$ and $Z$ in $V$ with $\mathrm{CS}(W, Z) > 0$), and that these path components are the $W$-$Z$-**islands** in $\mathrm{Ray}(V)$, having proved that $Z^\uparrow$ is the disjoint union of all $W$-$Z$-islands and the set $\{X \in Z^\uparrow \mid \check{X} = \varnothing\}$, and that, for any two intervals $A, B$ and rays $T_1 \in A$, $T_2 \in B$, the interval $[T_1, T_2]$ has glen $[T_{12}, T_{21}]$ in the "deep sea"

$$\mathrm{Ray}(V) \backslash Z^\uparrow = \{X \in \mathrm{Ray}(V) \mid \mathrm{CS}(W, X) \leqslant \mathrm{CS}(W, Z)\}$$

while $[T_1, T_{12}[ \subset A$, $]T_{21}, T_2] \subset B$.

A further study is needed to describe the sets of $W$-$Z$-islands which constitute the $Z$-equivalence classes in $Z^\uparrow$ different from the useless class of non-polar rays. This study is left for a future work.

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