

University of Aberdeen

Stabilization of the homotopy groups of the self equivalences of linear spheres

Libman, Assaf

Publication date:
2019

Document Version
Early version, also known as pre-print

[Link to publication](#)

Citation for published version (APA):
Libman, A. (2019). *Stabilization of the homotopy groups of the self equivalences of linear spheres*. ArXiv. <http://arxiv.org/abs/1903.12550v1>

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- ? Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- ? You may not further distribute the material or use it for any profit-making activity or commercial gain
- ? You may freely distribute the URL identifying the publication in the public portal ?

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

STABILIZATION OF THE HOMOTOPY GROUPS OF THE SELF EQUIVALENCES OF LINEAR SPHERES

ASSAF LIBMAN

ABSTRACT. Let G be a finite group. Let U_1, U_2, \dots be a sequence of orthogonal representations in which any irreducible representation of $\bigoplus_{n \geq 1} U_n$ has infinite multiplicity. Let $V_n = \bigoplus_{i=1}^n U_i$ and $S(V_n)$ denote the linear sphere of unit vectors. Then for any $i \geq 0$ the sequence of group $\dots \rightarrow \pi_i \operatorname{map}^G(S(V_n), S(V_n)) \rightarrow \pi_i \operatorname{map}^G(S(V_{n+1}), S(V_{n+1})) \rightarrow \dots$ stabilizes with the stable group $\bigoplus_H \omega_i(BW_G H)$ where H runs through representatives of the conjugacy classes of all the isotropy group of the points of $S(\bigoplus_n U_n)$.

1. INTRODUCTION

Let U be a real representation of a finite group G . Then U can be equipped with an essentially unique G -invariant norm. The set $S(U)$ of unit vectors is called a *linear sphere* for G . The one point compactification of U is denoted S^U .

This paper grew out of the interest in the homotopy groups of the space of equivariant self maps $\operatorname{End}^G(S(U)) \stackrel{\text{def}}{=} \operatorname{map}^G(S(U), S(U))$ as the dimension of U increases. To make this precise, observe that if U, V are orthogonal representations of G then $S(U \oplus V)$ is homeomorphic to the join $S(U) * S(V)$. More details about the join can be found in Section 4. Hence, if U_1, U_2, \dots is a sequence of (orthogonal) representations of G we obtain maps

$$\dots \rightarrow \operatorname{End}^G(S(\bigoplus_{i=1}^n U_i)) \xrightarrow{f \mapsto f * \operatorname{id}_{S(U_{n+1})}} \operatorname{End}^G(S(\bigoplus_{i=1}^{n+1} U_i)) \rightarrow \dots$$

In the spirit of the recent interest in “homological stability”, we ask whether these maps become isomorphism on homotopy groups for sufficiently large n . In other words, we seek a generalization of Freudenthal’s theorem to the equivariant setting (indeed, $\Sigma X \cong X * S^0$).

A map of unpointed spaces $f: X \rightarrow Y$ is called a *k-equivalence* (Definition 2.1) if it induces a bijection on path components, isomorphisms $\pi_i X \rightarrow \pi_i Y$ for all $1 \leq i \leq k$ and an epimorphism $\pi_{k+1} X \rightarrow \pi_{k+1} Y$ for any choice of basepoint in X . Let $\omega_i(X)$ denote the stable homotopy groups of X_+ (the disjoint union of X with a basepoint). Let BG denote the classifying space of a group G . Let (H) denote the conjugacy class of $H \leq G$ and $WH = N_G(H)/H$. If G acts on a space X , set $\operatorname{Iso}_G(X) = \{G_x : x \in X\}$ (isotropy groups).

Theorem 1.1. *Let U_1, U_2, \dots be real representations of a finite group G . Assume that if an irreducible representation R appears in one of the representation U_i then it appears in infinitely many of them. For any $n \geq 1$ set $X_n = S(U_1 \oplus \dots \oplus U_n)$. Let $k \geq 0$. Then for all sufficiently large n the maps*

$$\operatorname{map}^G(X_n, X_n) \xrightarrow{f \mapsto f * S(U_{n+1})} \operatorname{map}^G(X_{n+1}, X_{n+1})$$

are *k-equivalences*. In addition, for every $0 \leq i \leq k$ and any basepoint, the “stable groups” are isomorphic to (bijection if $i = 0$)

$$\pi_i \operatorname{map}^G(X_n, X_n) \cong \bigoplus_{(H) \subseteq \mathcal{F}} \omega_i(BWH)$$

where \mathcal{F} is the smallest collection of subgroups of G which is closed to intersections and contains $\operatorname{Iso}_G(S(V))$ for all the irreducible summands V of U_1, U_2, \dots .

2010 Mathematics Subject Classification. 55P91, 55Q52, 55P42.

Key words and phrases. Equivariant homotopy theory, homotopy groups, homotopy stabilization.

We point out that the challenge lies in the fact that the spaces X_n are not assumed to contain fixed points. If this is the case, namely if U_n contains the trivial representation for some n , Theorem 1.1 can be deduced from Hauschild's results [3, Satz 2.4]. Spectral sequence arguments were used by Schultz [8, Prop. 6.5] to prove Theorem 1.1 when G is cyclic. Becker and Schultz proved the theorem in the case that G acts freely [1] using geometric methods. Klaus proved that the groups $\pi_{k \geq 1} \text{End}^G(S(\oplus_{i=1}^n V))$ are finite for sufficiently large n and the identity map as basepoint. Using spectral sequence arguments the author improved this result in [6], showing that for each k the order of these groups become bounded as $n \rightarrow \infty$. However, spectral sequences become unmanageable in connection to the question of the stabilization of the homotopy groups.

The approach we take in this paper is different from earlier work. We use homotopy theoretic methods rather than geometric or algebraic methods. This allows us to separate the question of the stabilization from that of the identification of the stable groups. This is, again, quite different from the approach in previous work by the author and others.

Acknowledgements: I would like to thank Michael Crabb who read early versions of this paper and shared his ideas and most importantly, suggesting the description of the limit groups in Theorem 1.1 as a direct sum of stable homotopy groups.

Theorem 1.1 follows from Propositions 6.1 and 7.3 which can be viewed as the main technical results of this paper. The first is a general stabilization statement of the homotopy groups of joins of G -spaces and the second is a general result to identify the limit groups as stable homotopy groups. We decided to present the proof of Theorem 1.1 here to demonstrate how this works.

Proof of Theorem 1.1. The (unreduced) join of G -spaces is denoted $X * Y$ or simply XY . The join has two natural topologies [7, Section 2] but if X and Y are compact Hausdorff they both agree with the quotient topology of $(X \times Y \times I) \amalg X \amalg Y \rightarrow X * Y$. The action of G on the join is induced from its action on X, Y and $X \times Y \times I$. For compact Hausdorff spaces there is a homeomorphism $(XY)Z \cong X(YZ)$ and we may consider the n -fold join $X_1 \cdots X_n$ of G -spaces. See Section 4 for more details.

If $[n]$ denotes $\{1, \dots, n\}$, it follows that

$$(1.1) \quad \text{Iso}_G(X_1 * \cdots * X_n) = \bigcup_{\emptyset \neq A \subseteq [n]} \left\{ \bigcap_{i \in A} H_i : H_i \in \text{Iso}_G(X_i) \right\}.$$

If X and Y are CW-complexes then $\dim X * Y = \dim X + \dim Y + 1$. The dimension of the empty set is by convention -1 , and this fits well with the dimension formula just mentioned since $\emptyset * X = X$.

Let U_1, U_2, \dots be representations of G as in the statement of the theorem and $\mathcal{R} = \bigcup_n \text{Irr}(U_n)$ the set of their irreducible summands. Let \mathcal{F} be the smallest collection of subgroups of G which contains $\text{Iso}_G(S(V))$ for all $V \in \mathcal{R}$, and is closed to intersections. There are homeomorphisms $S(U_1 \oplus \cdots \oplus U_n) \cong S(U_1) * \cdots * S(U_n)$ and it is clear from (1.1) and the finiteness of G that there is a representation V' such that $\text{Irr}(V') = \mathcal{R}$ and $\text{Iso}_G(S(V')) = \mathcal{F}$. Then $\dim V'^H \geq 1$ for any $H \in \mathcal{F}$, and if $K \succeq H$ then $\dim V'^H - \dim V'^K \geq 1$ since there are $x \in V'^H$ with $G_x = H$ so $V'^K \subsetneq V'^H$. Given $k \geq 0$, taking $k+1$ copies of V' shows that there exists a representation V such that

$$(1.2) \quad \begin{aligned} \text{Irr}(V) &\subseteq \mathcal{R}, \text{ and } \text{Iso}_G(V) = \mathcal{F}, \\ \dim S(V)^H &\geq k \text{ for any } H \in \mathcal{F}, \text{ and} \\ \dim S(V)^H - \dim \bigcup_{K \succeq H} S(V)^K &\geq k. \end{aligned}$$

(the last inequality needs to be proven separately for the case $H \preceq G$ and for $H = G$).

Proof of the stabilization: Fix $k \geq 0$. The hypothesis on U_1, U_2, \dots and (1.2) imply that there exist integers $n_0 \geq m \geq 1$ such that $X \stackrel{\text{def}}{=} S(U_1 \oplus \dots \oplus U_m)$ and $Y' \stackrel{\text{def}}{=} S(U_{m+1} \oplus \dots \oplus U_{n_0})$ have the property that

$$\text{Iso}_G(X) = \text{Iso}_G(Y') = \mathcal{F}$$

and for any $H \in \mathcal{F}$

$$\dim X^H - \dim \bigcup_{K \geq H} X^K \geq k + 1 \quad \text{and} \quad \dim Y'^H \geq k + 1.$$

Let $n \geq n_0$ and set $Y = S(U_{m+1} \oplus \dots \oplus U_n)$ and $Z = S(U_{n+1})$. With the notation of the theorem, $X_n = X * Y$ and $X_{n+1} = X * Y * Z$ and the first statement of the theorem is that

$$\text{map}^G(X * Y, X * Y) \xrightarrow{f \mapsto f * \text{id}_Z} \text{map}^G(X * Y * Z, X * Y * Z)$$

is a k -equivalence. To prove this we apply Proposition 6.1. First, X, Y and Z are finite G -CW complexes by Illman's result [4]. Since $Y' \subseteq Y$, the choice of \mathcal{F} and Y' guarantees that $\text{Iso}(X) = \text{Iso}(Y) = \text{Iso}(X * Y * Z) = \mathcal{F}$, so hypothesis (1) of Proposition 6.1 holds. Hypothesis (2) also holds since $\dim Y'^H \geq \dim Y^H \geq k + 1$ for all $H \in \mathcal{F}$, and our choice of X satisfies hypothesis (3). Now, $(X * Y)^H \cong S(U_1 \oplus \dots \oplus U_n)^H$ is itself a linear sphere of dimension $\dim X^H + \dim Y^H + 1$, hence it is a $(\dim X^H + \dim Y^H)$ -connected space. Similarly $(X * Y * Z)^H$ is a linear sphere of dimension $\dim X^H + \dim(Y * Z)^H + 1$ and hypothesis (4) also holds. Now, Z^H is a sphere and the join of a space A with $S^m \cong S^0 * \dots * S^0$ is homeomorphic to the $(m + 1)$ -fold unreduced suspension of A . An iterated use of Proposition 2.6 shows that the map $F(X^H, (X * Y)^H) \rightarrow F(X^H * Z^H, (X * Y)^H * Z^H)$, where $F(-, -)$ denotes the space of (non-equivariant) continuous maps, is a $(\dim(XY)^H - 1)$ -equivalence. Hypothesis (5) of Proposition 6.1 holds since $\dim(XY)^H - 1 = \dim X^H + \dim Y^H \geq \dim X^H + k + 1$.

Calculation of the limit groups: The hypothesis on U_1, U_2, \dots together with (1.2) guarantee that if n is large enough, then $X = X_n = S(U_1 \oplus \dots \oplus U_n)$ has the property that $\text{Iso}_G(X) = \mathcal{F}$ and that $\dim X^H \geq k + 2$ and $\dim X^H - \dim \bigcup_{K \geq H} X^K \geq k + 2$ for any $H \in \mathcal{F}$. It remain to show that $\pi_i \text{map}^G(X, X)$ are isomorphic to those in the statement of the theorem for $0 \leq i \leq k$.

This follows from Proposition 7.3 which we proceed to check its hypotheses (a)–(d). Set $U = U_1 \oplus \dots \oplus U_n$. Clearly, $\dim X = \dim U - 1$ which is hypothesis (a). Hypothesis (b) was shown above. Also, $S^U \setminus S(U)$ is clearly WH -equivalent to S^0 and hypothesis (c) holds. We observe that $\Sigma S(U) \cong S^U$, the unreduced suspension. The adjoint of this homeomorphism is a map $\eta: S(U) \rightarrow \mathbb{P}_{\alpha_0, \alpha_1} S^U$ where α_0, α_1 are the points $0 \in U \subseteq S^U$ and $\infty \in S^U$ and $\mathbb{P}_{\alpha_0, \alpha_1} S^U$ is the space of paths $I \rightarrow S^U$ which send $0 \mapsto \alpha_0$ and $1 \mapsto \alpha_1$. For any $H \in \mathcal{F}$, taking fixed points gives a non equivariant map $\eta^H: S(U^H) \rightarrow \mathbb{P}_{\alpha_0, \alpha_1} \Sigma S(U^H)$ which is homotopic to the natural (non-equivariant) map $S(U^H) \rightarrow \Omega \Sigma S(U^H)$ and is therefore a $(2 \dim S(U)^H - 2)$ -equivalence by Freudenthal's theorem. Now, $S(U)^H = X^H$ and $2 \dim X^H - 2 \geq \dim X^H + k$ because $\dim X^H \geq k + 1$, so hypothesis (d) holds. The result now follows from (7.2) of Proposition 7.3 because $EW H_+ \wedge_{WH} S^0 \simeq BWH_+$. \square

2. PRELIMINARIES

Definition 2.1. A map $f: X \rightarrow Y$ of spaces is called a k -equivalence if it is bijective on components and for any $x \in X$ the induced maps $\pi_i(X, x) \rightarrow \pi_i(Y, f(x))$ are isomorphisms for all $1 \leq i \leq k$ and epimorphism for $k + 1$.

This is just a (convenient) “shift by 1” of the standard definitions of n -connectedness of maps, see [12]. A space X is called k -connected, where $k \geq 0$, if it is k -equivalent to a point. We will write

$$\text{conn } X = k.$$

By convention $\text{conn } X = -1$ if the number of path components of X is not 1. The next two results are straightforward.

Lemma 2.2. *Let $k \geq 0$. Consider a morphism of fibre sequences where $b_1 \in B_1$.*

$$\begin{array}{ccccc} \text{Fib}(p_1, b_1) & \longrightarrow & E_1 & \xrightarrow{p_1} & B_1 \\ \downarrow h & & \downarrow g & & \downarrow f \\ \text{Fib}(p_2, f(b_1)) & \longrightarrow & E_2 & \xrightarrow{p_2} & B_2. \end{array}$$

- (1) *If f is a k -equivalence and the map h is a k -equivalence for any choice of $b_1 \in B_1$ then g is a k -equivalence.*
- (2) *If f is a $(k+1)$ -equivalence and g is a k -equivalence then h is a k -equivalence for any choice of $b_1 \in B_1$.*

Proof. This is standard diagram chase of exact sequences of pointed sets and groups. The first assertion is slightly more delicate in connection to the surjection on components and uses the homotopy lifting property of p_2 . \square

Lemma 2.3. *Let $k \geq 0$ and consider the following ladder of Serre fibrations*

$$\begin{array}{ccccc} E_1 & \xrightarrow{q_1} & D_1 & \xrightarrow{p_1} & B_1 \\ h \downarrow & & \downarrow g & & \downarrow f \\ E_2 & \xrightarrow{q_2} & D_2 & \xrightarrow{p_2} & B_2 \end{array}$$

Assume that f, g, h induce k -equivalences on the fibres of p_1 and p_2 and on the fibres of q_1 and q_2 . Then they induce k -equivalences on the fibres of $p_1 \circ q_1$ and $p_2 \circ q_2$.

Proof. Choose $b_1 \in B_1$ and set $b_2 = f(b_1)$. We need to show that $F_i = \text{Fib}(p_i \circ q_i, b_i)$ are k -equivalent. Set $X_i = \text{Fib}(p_i, b_i)$. We obtain a morphism of fibrations

$$\begin{array}{ccc} F_1 & \xrightarrow{q_1|_{F_1}} & X_1 \\ h|_{F_1} \downarrow & & \downarrow g|_{X_1} \\ F_2 & \xrightarrow{q_2|_{F_2}} & X_2 \end{array}$$

By hypothesis $g|_{X_1}$ is a k -equivalence, so by Lemma 2.2(1) it remain to show that all the fibres of the rows of this diagram are k -equivalent. These fibres are equal to the fibres of q_1 and q_2 and by the hypothesis they are k -equivalent. \square

Throughout this paper we will work in a ‘‘convenient category of G -spaces’’, that is the category \mathcal{CGWH} of compactly generated weak Hausdorff spaces, or the category \mathcal{CGH} of compactly generated Hausdorff spaces, see [9] or [10]. This category has products and function complexes $F(X, Y)$ giving adjunction homeomorphisms $\text{map}(Z \times X, Y) \cong \text{map}(Z, F(X, Y))$ where map denotes the set of morphisms in \mathcal{CGWH} . In fact, \mathcal{CGWH} is enriched over itself and $\text{map}(X, Y) \cong F(X, Y)$.

Let G be a discrete group, e.g finite. Let $G\text{-}\mathcal{CGWH}$ be the category of G -spaces. Regarding X and Y as objects in \mathcal{CGH} via the forgetful functor, $F(X, Y)$ is equipped with a standard action of G where $(g \cdot \varphi)(x) = g\varphi(g^{-1}x)$. In this way the set of all G -maps from X to Y denoted $\text{map}^G(X, Y)$ is equipped with a topology giving rise to the adjunction homeomorphism $\text{map}^G(Z \times X, Y) \cong \text{map}^G(Z, F(X, Y))$.

If $B \subseteq A$ and A is obtained from B by attaching equivariant cells (or more generally, if $B \subseteq A$ is a G -cofibration) then $\text{map}^G(A, X) \rightarrow \text{map}^G(B, X)$ is a Serre fibration for any G -space X .

Such an inclusion $B \subseteq A$ is called a *relative G -CW complex*. A G -CW complex is a space obtained in this way from the empty set. See [11, Chapter II.1]. We emphasize that by G -CW complexes we always mean that G acts cellularly (by permuting cells).

Let $Y \subseteq X$ be an inclusion of G -subspaces. We denote by G_x the stabilizer of $x \in X$. Set

$$\text{Iso}_G(X, Y) = \{G_x : x \in X \setminus Y\}.$$

If $Y = \emptyset$ we will simply write $\text{Iso}_G(X)$.

Let $B \subseteq A$ be an inclusion of finite dimensional G -CW complexes (see e.g [11, Chap. II]). For any $H \in \text{Iso}_G(A)$ let

$$\dim_H(A, B)$$

denote the maximum dimension of an equivariant cell of type G/H in A which is not contained in B . Thus, $\dim_H(A, B) = \dim(A^H, B^H)$, the relative dimension of the inclusion of CW-complexes $B^H \subseteq A^H$.

A map of G -spaces is called a k -equivalence if this is the case by forgetting the action of G .

Lemma 2.4. *Fix some $k \geq 0$. Suppose $B \subseteq A$ is an inclusion of finite dimensional G -CW complexes and $f: X \rightarrow Y$ is a map of G -spaces. For any $H \in \text{Iso}_G(A, B)$ set $n_H = \dim_H(A, B)$ and assume that the map $X^H \xrightarrow{f^H} Y^H$ is a $(k + n_H)$ -equivalence. Then for any $\varphi \in \text{map}^G(B, X)$ the map f_* induced on fibres in*

$$\begin{array}{ccccc} \text{Fib}(i^*, \varphi) & \longrightarrow & \text{map}^G(A, X) & \xrightarrow{i^*} & \text{map}^G(B, X) \\ f_* \downarrow & & f_* \downarrow & & \downarrow f_* \\ \text{Fib}(i^*, f \circ \varphi) & \longrightarrow & \text{map}^G(A, Y) & \xrightarrow{i^*} & \text{map}^G(B, Y) \end{array}$$

is a k -equivalence of spaces.

Proof. We use induction on $n = \dim(A, B)$. If $n = -1$ then $A = B$ and the result is trivial. Assume that $n \geq 0$ and let $A' \subseteq A$ be the union of the $(n - 1)^{\text{st}}$ skeleton of A with B . Let $i: B \rightarrow A$ and $j: B \rightarrow A'$ and $\ell: A' \rightarrow A$ denote the inclusions. We obtain a diagram of fibrations

$$(2.1) \quad \begin{array}{ccccc} \text{map}^G(A, X) & \xrightarrow{\ell^*} & \text{map}^G(A', X) & \xrightarrow{j^*} & \text{map}^G(B, X) \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ \text{map}^G(A, Y) & \xrightarrow{\ell^*} & \text{map}^G(A', Y) & \xrightarrow{j^*} & \text{map}^G(B, Y) \end{array}$$

such that composition of the rows are the maps i^* . By construction $\dim(A', B) \leq n - 1$ and also $\dim_H(A', B) \leq \dim_H(A, B) = n_H$ for any $H \in \text{Iso}_G(A', B)$. The induction hypothesis applies to the inclusion $B \subseteq A'$ and we deduce that the fibres of j^* in the second square are k -equivalent. By Lemma 2.3 it remains to show that the fibres of the maps ℓ^* are k -equivalent.

Since A is obtained from A' by attaching equivariant n -cells, we get a pushout diagram

$$\begin{array}{ccc} \coprod_{i \in \mathcal{A}} S^{n-1} \times G/H_i & \xrightarrow{\eta} & A' \\ \downarrow & & \downarrow \\ \coprod_{i \in \mathcal{A}} D^n \times G/H_i & \longrightarrow & A. \end{array}$$

where \mathcal{A} indexes the equivariant n -cells attached to A' . Notice that $n_{H_i} = n$ for all $i \in \mathcal{A}$. By applying $\text{map}^G(-, T)$, where T is any G -space, we obtain a pullback diagram, natural in

T ,

$$\begin{array}{ccc} \text{map}^G(A, T) & \longrightarrow & \text{map}^G(A', T) \\ \downarrow & & \downarrow \eta^* \\ \prod_i \text{map}(D^n, T^{H_i}) & \longrightarrow & \prod_i \text{map}(S^{n-1}, T^{H_i}). \end{array}$$

Since this is a pullback square, the fibres of the horizontal maps are homeomorphic. Applying this for $T = X$ and $T = Y$, it remains to prove that the maps on all fibres in the following commutative diagram

$$\begin{array}{ccc} \prod_i \text{map}(D^n, X^{H_i}) & \longrightarrow & \prod_i \text{map}(S^{n-1}, X^{H_i}) \\ \downarrow f_* & & \downarrow f_* \\ \prod_i \text{map}(D^n, Y^{H_i}) & \longrightarrow & \prod_i \text{map}(S^{n-1}, Y^{H_i}) \end{array}$$

are k -equivalences. If $n = 0$ then the spaces on the right are points and the map of the spaces on the left is by hypothesis $n_{H_i} + k = n + k = k$ equivalence, so the map on fibres is a k -equivalence. If $n \geq 1$ then by the hypothesis the vertical arrow on the left is a $(k+n)$ -equivalence, hence a k -equivalence, and the vertical arrow on the right is a $k+n-(n-1) = k+1$ equivalence. Lemma 2.2 shows that the map on all fibres is a k -equivalence and this completes the proof. \square

Corollary 2.5. *Let $k \geq 0$. Let A be a finite G -CW complex and B be a subcomplex. Let Y be a G space and let $k \geq 0$. Assume that $\text{conn } Y^H - \dim_H(A, B) \geq k$ for every $H \in \text{Iso}_G(A, B)$. Then $\text{map}^G(A, Y) \xrightarrow{i^*} \text{map}^G(B, Y)$ is a k -equivalence.*

Proof. Lemma 2.4 with $B \subseteq A$ and $Y \rightarrow *$ shows that all the fibres of $\text{map}^G(A, Y) \rightarrow \text{map}^G(B, Y)$ are k -connected. Then apply Lemma 2.2(1) with $g = i^*$ and f the identity on $\text{map}^G(B, Y)$. \square

We will need the following simple corollary of Freudenthal's theorem.

Proposition 2.6. *Let $m > n \geq 1$ and let susp denote the unreduced suspension functor. Then the map $F(S^n, S^m) \xrightarrow{\text{susp}} F(\Sigma S^n, \Sigma S^m)$ is a $(m-1)$ -equivalence of path connected spaces.*

Proof. Recall that ΣX is the quotient of $I \times X$ obtained by collapsing each one of the subspaces $\{0\} \times X$ and $\{1\} \times X$ to a point. If (X, x_0) is a based space, we will choose $(\frac{1}{2}, x_0)$ as the basepoint of ΣX . There is an inclusion $\text{incl}: X \rightarrow \Sigma X$ via $x \mapsto (\frac{1}{2}, x)$. The reduced suspension SX is obtained by further collapsing $I \times \{x_0\}$ to a point. If X is a pointed CW-complex then the quotient map $q: \Sigma X \rightarrow SX$ is a homotopy equivalence.

Choose once and for all basepoints in S^n and S^m . We will write S^{n+1} and S^{m+1} for their reduced suspensions. We obtain the following diagram where ev are the evaluation fibrations.

$$\begin{array}{ccccccc} F_*(S^n, S^m) & \xrightarrow{\text{susp}_*} & & & F_*(S^{n+1}, S^{m+1}) & & \\ \downarrow & & & & \downarrow & & \\ F(S^n, S^m) & \xrightarrow{\text{susp}} & F(\Sigma S^n, \Sigma S^m) & \xrightarrow{q_*} & F(\Sigma S^n, S^{m+1}) & \xleftarrow{q^*} & F(S^{n+1}, S^{m+1}) \\ \downarrow \text{ev} & & \downarrow \text{ev} & & \downarrow \text{ev} & & \downarrow \text{ev} \\ S^m & \xrightarrow{\text{incl}} & \Sigma S^m & \xrightarrow{q} & S^{m+1} & \xlongequal{\quad} & S^{m+1} \end{array}$$

The top square commutes because for any $f: X \rightarrow Y$ of pointed spaces Sf is the unique map such that $q_Y \circ \Sigma f = Sf \circ q_X$ where $q_X: \Sigma X \rightarrow SX$ and $q_Y: \Sigma Y \rightarrow SY$ are the quotient maps.

The maps q_* and q^* are homotopy equivalences since q are. The bottom squares commute by inspection.

By Freudenthal's theorem susp_* is a $(2m - 2)$ -equivalence, hence a $(m - 1)$ -equivalence since $m \geq 1$. In addition $\pi_i S^{m+1} = 0$ for $i \leq m$. Inspection of the morphism of the long exact sequences in homotopy groups of the fibre sequences at the left and right columns shows that $F(S^n, S^m) \rightarrow F(S^{n+1}, S^{m+1})$ in the second row of the diagram is a $(m - 1)$ -equivalence of path connected spaces. The result follows. \square

3. SQUARE DIAGRAMS OF SPACES

For a space X let $\mathbb{P}X$ denote the path space $\text{map}(I, X)$. The homotopy pullback of a diagram of spaces $X_0 \xrightarrow{f} X_2 \xleftarrow{g} X_1$ is the subspace of $X_0 \times X_1 \times \mathbb{P}X_2$ consisting of (x_0, x_1, ω) such that $f(x_0) = \omega(0)$ and $g(x_1) = \omega(1)$. The homotopy fibre of $X \xrightarrow{f} Y$ over $y_0 \in Y$ is the homotopy pullback of $X \xrightarrow{f} Y \xleftarrow{y_0} *$. There is an inclusion $\text{Fib}(f, y_0) \subseteq \text{hoFib}(f, y_0)$ via the constant paths, and if f is a Serre fibration this inclusion is a weak homotopy equivalence.

Definition 3.1. Let \mathfrak{S} be the category whose objects are commutative diagrams of spaces of the form

$$(3.1) \quad \mathcal{A} = \begin{array}{ccc} A_3 & \xrightarrow{a_{32}} & A_2 \\ a_{31} \downarrow & & \downarrow a_{20} \\ A_1 & \xrightarrow{a_{10}} & A_0 \end{array}$$

Morphisms are natural transformations of diagrams. Thus, a morphism $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ is a quadruple $(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ of maps $\varphi_i: A_i \rightarrow B_i$ with the obvious commutation relations with the structure maps of \mathcal{A} and \mathcal{B} .

A *basepoint* for \mathcal{A} is a triple of $\underline{x} = (x_0, x_1, x_2) \in A_0 \times A_1 \times A_2$ such that $a_{20}(x_2) = x_0$ and $a_{10}(x_1) = x_0$. Notice that we do not choose $x_3 \in A_3$ compatible with x_0, x_1, x_2 .

A basepoint of $\mathcal{A} \in \mathfrak{S}$ gives rise to the following diagram of spaces which we denote by $(\mathcal{A}, \underline{x})$.

$$(3.2) \quad (\mathcal{A}, \underline{x}) = \begin{array}{ccccc} A_3 & \xrightarrow{a_{32}} & A_2 & \xleftarrow{x_2} & \bullet \\ a_{31} \downarrow & & \downarrow a_{20} & & \downarrow \\ A_1 & \xrightarrow{a_{10}} & A_0 & \xleftarrow{x_0} & \bullet \\ x_1 \uparrow & & \uparrow & & \uparrow \\ \bullet & \longrightarrow & \bullet & \longleftarrow & \bullet \end{array}$$

We obtain a category \mathfrak{S}_* whose objects are $(\mathcal{A}, \underline{x})$ with natural transformations between them. The homotopy limit functor gives rise to a functor $\Lambda: \mathfrak{S}_* \rightarrow \mathbf{Spaces}$

$$(3.2) \quad \Lambda(\mathcal{A}, \underline{x}) \stackrel{\text{def}}{=} \text{holim}(\mathcal{A}, \underline{x}).$$

Fubini's theorem for homotopy limits [2, Secs. 24 and 31] implies that $(\mathcal{A}, \underline{x})$ can be calculated by first taking the homotopy limits of the rows (resp. columns) and then take the homotopy limits of the resulting pullback diagram of spaces. Since the homotopy limits of the rows (resp. columns) of $(\mathcal{A}, \underline{x})$ are the homotopy fibres of the rows of \mathcal{A} over x_2 and x_0 (resp. the homotopy fibres of the columns of \mathcal{A} over x_1 and x_0), it follows that

$$(3.3) \quad \begin{aligned} \Lambda(\mathcal{A}, \underline{x}) &\cong \text{hoFib} \left(\text{hoFib}(a_{32}, x_2) \xrightarrow{(a_{31}, a_{20})} \text{hoFib}(a_{10}, x_0), x_1 \right) \\ &\cong \text{hoFib} \left(\text{hoFib}(a_{31}, x_1) \xrightarrow{(a_{32}, a_{10})} \text{hoFib}(a_{20}, x_0), x_2 \right) \end{aligned}$$

where $x_1 \in \text{Fib}(a_{10}, x_0) \subseteq \text{hoFib}(a_{10}, x_0)$ and $x_2 \in \text{Fib}(a_{20}, x_0) \subseteq \text{hoFib}(a_{20}, x_0)$.

Lemma 3.2. *Let $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ be a morphism in \mathfrak{S} depicted by the vertical arrows in*

$$\begin{array}{ccccc}
 A_3 & \xrightarrow{a_{32}} & A_2 & & \\
 \varphi_3 \downarrow & \searrow & \downarrow & \searrow^{a_{20}} & \\
 & & A_1 & \xrightarrow{\quad} & A_0 \\
 & & \downarrow & & \downarrow \varphi_0 \\
 B_3 & \xrightarrow{\quad} & B_2 & & \\
 b_{31} \searrow & & \downarrow & \searrow & \\
 & & B_1 & \xrightarrow{b_{10}} & B_0
 \end{array}$$

- (1) *If the side faces (or the back and front faces) are homotopy pullback squares then the induced map $\Lambda(\mathcal{A}, \underline{x}) \xrightarrow{\Lambda(\varphi)} \Lambda(\mathcal{B}, \varphi(\underline{x}))$ is a (weak) homotopy equivalence for any choice of basepoint \underline{x} for \mathcal{A} .*
- (2) *Let \underline{x} be a basepoint for \mathcal{A} . Suppose that*
- (i) φ_2 and φ_0 *induce a $(k+1)$ -equivalence $\text{hoFib}(a_{20}, x_0) \rightarrow \text{hoFib}(b_{20}, \varphi_0(x_0))$ and*
 - (ii) φ_1, φ_3 *induce a k -equivalence $\text{hoFib}(a_{31}, x_1) \rightarrow \text{hoFib}(b_{31}, \varphi_1(x_1))$.*
- Then $\Lambda(\mathcal{A}, \underline{x}) \xrightarrow{\Lambda(\varphi)} \Lambda(\mathcal{B}, \varphi(\underline{x}))$ is a k -equivalence.*
- (3) *Suppose that $A_2 \xrightarrow{a_{20}} A_0$ is a Serre fibration and that*
- (i) $\text{hoFib}(a_{20}, x_0) \rightarrow \text{hoFib}(b_{20}, \varphi_1(x_0))$ *is a k -equivalence for any basepoint $x_0 \in A_0$, and*
 - (ii) $\Lambda(\mathcal{A}, \underline{x}) \xrightarrow{\Lambda(\varphi)} \Lambda(\mathcal{B}, \varphi(\underline{x}))$ *is a k -equivalence for any choice of basepoint \underline{x} in \mathcal{A} . Then $\text{hoFib}(a_{31}, x_1) \rightarrow \text{hoFib}(b_{31}, \varphi(x_1))$ is a k -equivalence for any basepoint $x_1 \in A_1$.*

Proof. (1) Since each side face is a homotopy pullback square, the induced maps on homotopy fibres of its rows is a weak equivalence. Thus, the vertical arrows in

$$(3.4) \quad \begin{array}{ccc}
 \text{hoFib}(a_{31}, x_1) & \xrightarrow{(a_{32}, a_{10})} & \text{hoFib}(a_{20}, x_0) \\
 (\varphi_3, \varphi_1) \downarrow & & \downarrow (\varphi_2, \varphi_0) \\
 \text{hoFib}(b_{31}, \varphi_1(x_1)) & \xrightarrow{(b_{32}, b_{10})} & \text{hoFib}(b_{20}, \varphi_0(x_0))
 \end{array}$$

are weak homotopy equivalences. Therefore the map induced on the homotopy fibres of the rows over x_2 and $\varphi_2(x_2)$ are weak equivalences, and the result follows from (3.3).

(2) This is immediate from (3.3) and Lemma 2.2(2).

(3) Choose $x_1 \in A_1$ and set $x_0 = a_{10}(x_1)$. There results a commutative diagram as in (3.4). By the hypothesis the vertical arrow on the right of (3.4) is a k -equivalence, and our goal is to show the the same is true for the vertical arrow on the left. By Lemma 2.2(1) it remain to show that for any $x_2 \in \text{hoFib}(a_{20}, x_0)$ the map induced on the homotopy fibres of the horizontal arrows is a k -equivalence. By hypothesis $A_2 \rightarrow A_0$ is a Serre fibration, so the inclusion $\text{fib}(a_{20}, x_0) \subseteq \text{hoFib}(a_{20}, x_0)$ is a weak homotopy equivalence. Therefore we may consider only $x_2 \in \text{fib}(a_{20}, x_0)$ in which case $\underline{x} = (x_0, x_1, x_2)$ forms a basepoint for \mathcal{A} and it follows from (3.3) that the map of the homotopy fibres over x_2 and $\varphi_2(x_2)$ in the diagram (3.4) is the map $\Lambda(\mathcal{A}, \underline{x}) \xrightarrow{\Lambda(\varphi)} \Lambda(\mathcal{B}, \varphi(\underline{x}))$ which by hypothesis is a k -equivalence. This completes the proof. \square

Given an object $\mathcal{A} \in \mathfrak{S}$ let $I \times \mathcal{A}$ be the object in \mathfrak{S} obtained by applying the functor $I \times -$ objectwise. Similarly $\mathbb{P}\mathcal{A}$ is the object in \mathfrak{S} obtained by applying the path space functor $\mathbb{P}(-)$ objectwise.

Definition 3.3. Let \mathcal{A}, \mathcal{B} be objects in \mathfrak{S} . A *homotopy* is a morphism $\varphi: I \times \mathcal{A} \rightarrow \mathcal{B}$. We frequently refer to a homotopy as a family of morphisms $\varphi_p: \mathcal{A} \rightarrow \mathcal{B}$ (parametrized by $0 \leq p \leq 1$).

The adjoint of a homotopy $\varphi: I \times \mathcal{A} \rightarrow \mathcal{B}$ is a morphism $\varphi^\#: \mathcal{A} \rightarrow \mathbb{P}\mathcal{B}$. If \underline{x} is a basepoint of \mathcal{A} then $\varphi^\#(\underline{x})$ is a basepoint in $\mathbb{P}\mathcal{B}$. Evaluation at $p \in I$ gives a morphism $(\mathbb{P}\mathcal{B}, \varphi^\#(\underline{x})) \xrightarrow{\text{ev}_p} (\mathcal{B}, \varphi_p(\underline{x}))$ in \mathfrak{S}_* which is an object-wise homotopy equivalence. We obtain a weak homotopy equivalence $\Lambda(\mathbb{P}\mathcal{B}, \varphi^\#(\underline{x})) \xrightarrow{\Lambda(\text{ev}_p)} \Lambda(\mathcal{B}, \varphi_p(\underline{x}))$. The following lemma is an immediate consequence.

Lemma 3.4. *Let $\varphi_p: \mathcal{A} \rightarrow \mathcal{B}$ be a homotopy in \mathfrak{S} . Then for any basepoint \underline{x} in \mathcal{A} there is a commutative diagram in which both evaluation morphisms are (weak) homotopy equivalences*

$$\begin{array}{ccc} \Lambda(\mathcal{A}, \underline{x}) & \xrightarrow{\Lambda(\varphi_{p=0})} & \Lambda(\mathcal{B}, \varphi_{p=0}(\underline{x})) \\ \Lambda(\varphi_{p=1}) \downarrow & \searrow \Lambda(\varphi^\#) & \simeq \uparrow \Lambda(\text{ev}_0) \\ \Lambda(\mathcal{B}, \varphi_{p=1}(\underline{x})) & \xleftarrow[\Lambda(\text{ev}_1)]{\simeq} & \Lambda(\mathbb{P}\mathcal{B}, \varphi^\#(\underline{x})) \end{array}$$

In particular, if $\alpha, \beta: \mathcal{A} \rightarrow \mathcal{B}$ are homotopic morphisms and \underline{x} of \mathcal{A} is a basepoint then $\Lambda(\alpha, \underline{x})$ is a k -equivalence if and only if $\Lambda(\beta, \underline{x})$ is.

4. JOIN OF SPACES

The join of spaces X_1, \dots, X_n denoted $X_1 * \dots * X_n$ is the homotopy colimit of the diagram of spaces indexed by the poset of the non-empty subsets σ of $[n] = \{1, \dots, n\}$ consisting of the spaces $X_\sigma \stackrel{\text{def}}{=} \prod_{i \in \sigma} X_i$ and projection maps between them. Let $\Delta^{n-1} = \{(t_1, \dots, t_n) : t_i \geq 0, \sum_i t_i = 1\}$ be the standard $(n-1)$ -simplex in \mathbb{R}^n . The underlying set of the join is the set of equivalence classed of

$$(4.1) \quad \coprod_{\emptyset \neq \sigma \subseteq [n]} \Delta^{|\sigma|-1} \times X_\sigma$$

where for any $\tau \subseteq \sigma$ we declare $(s_i, x_i)_{i \in \tau} \sim (t_i, y_i)_{i \in \sigma}$ if $x_i = y_i$ and $s_i = t_i$ for all $i \in \tau$ and $t_i = 0$ for all $i \in \sigma \setminus \tau$. There are two natural choices to topologize the join, but when the spaces X_1, \dots, X_n are compact Hausdorff both agree with the quotient topology, see [7, Section 2] (and notice that when the spaces X_i are compact Hausdorff the weak and strong topologies on the join coincide). In light of all this, we will use the following notation throughout.

4.1. NOTATION: Since the join will play a key role in this paper we will write

$$X_1 \cdots X_n \quad \text{instead of} \quad X_1 * \dots * X_n.$$

Its points are equivalence classes $[t_1 x_1, \dots, t_n x_n]$ where $(t_1, \dots, t_n) \in \Delta^{n-1}$ and it is understood that $t_i x_i$ may be omitted from the notation if either X_i is empty or if $t_i = 0$, and two such brackets represent the same point if they agree except in the entries where $t_i = 0$.

Identify Δ^1 with the unit interval I via the homeomorphism $I \xrightarrow{t \mapsto (t, 1-t)} \Delta^1$. Then the join XY of compact Hausdorff spaces is the quotient space of $X \times Y \times I \amalg X \amalg Y$ under the equivalence relation generated by $(x, y, 0) \sim y$ and $(x, y, 1) \sim x$. Thus, there is a pushout square

$$(4.2) \quad \begin{array}{ccc} (X \times Y) \amalg (X \times Y) & \xrightarrow{c^{i_0+i_1}} & X \times Y \times I \\ \pi_Y \amalg \pi_X \downarrow & & \downarrow \pi \\ Y \amalg X & \xrightarrow{c} & X * Y \end{array}$$

If X_1, \dots, X_n are G -spaces then their join is also a G -space via the diagonal action. Given a G -space Z we obtain a functor $X \mapsto XZ$ from $G\text{-CGWH}$ to itself. This functor is, in fact,

continuous in the sense that for any G -spaces X, Y the resulting natural map

$$(4.3) \quad J_Z: F(X, Y) \xrightarrow{f \mapsto f * \text{id}_Z} F(XZ, YZ)$$

is a continuous map. We can describe it explicitly: for any $f \in F(X, Y)$

$$(4.4) \quad J_Z(f)([sx, (1-s)z]) = [s \cdot f(x), (1-s)z].$$

One easily checks that J_Z is G -equivariant and passage to fixed points gives

$$J_Z: \text{map}^G(X, Y) \xrightarrow{f \mapsto f * \text{id}_Z} \text{map}^G(XZ, YZ)$$

If X, Y, Z are compact there are well known natural ‘‘associativity’’ homeomorphisms

$$(4.5) \quad \begin{aligned} (XY)Z &\xrightarrow{[s[tx, (1-t)y], (1-s)z] \mapsto [stx, s(1-t)y, (1-s)z]} XYZ \\ X(YZ) &\xrightarrow{[tx, (1-t)[sy, (1-s)z]] \mapsto [tx, s(1-t)y, (1-s)(1-t)z]} XYZ \end{aligned}$$

This allows us to identify, for example, $\text{map}^G(AY, XY) \xrightarrow{J_Z} \text{map}^G((AY)Z, (XY)Z)$ with $\text{map}^G(AY, XY) \xrightarrow{J_Z} \text{map}^G(AYZ, XYZ)$

By inspection, these homeomorphisms together with (4.4) imply the commutativity of the following diagrams for compact G -spaces A, T, Y, Z .

$$(4.6) \quad \begin{array}{ccc} \text{map}^G(A, T) & \xrightarrow{J_Z} & \text{map}^G(AZ, TZ) \\ & \searrow \text{incl}_* & \downarrow i^* \\ & & \text{map}^G(A, TZ) \\ \text{map}^G(AY, T) & \xrightarrow{i_Y^*} & \text{map}^G(Y, T) \\ \downarrow J_Z & & \downarrow J_Z \\ \text{map}^G(AYZ, TZ) & \xrightarrow{i_{YZ}^*} & \text{map}^G(YZ, TZ) \end{array}$$

where $\text{incl}: T \xrightarrow{t \mapsto [1-t, 0]} TZ$ and $i: A \xrightarrow{a \mapsto [1-a, 0]} AZ$ are the inclusions and we used the homeomorphism $(AY)Z \cong AYZ$.

Definition 4.2. Let A, X, Y, Z be compact G -spaces. Let

$$\psi_{A, X, Y, Z}: \text{map}^G(A \times X, Y) \rightarrow \text{map}^G(A \times XZ, YZ)$$

be the unique map which renders the following diagram commutative

$$\begin{array}{ccc} \text{map}^G(A, F(X, Y)) & \xrightarrow{\text{map}^G(A, J_Z)} & \text{map}^G(A, F(XZ, YZ)) \\ \cong \downarrow & & \downarrow \cong \\ \text{map}^G(A \times X, Y) & \xrightarrow{\psi_{A, X, Y, Z}} & \text{map}^G(A \times XZ, YZ). \end{array}$$

It is clear that ψ is natural in A . By inspection

$$(4.7) \quad \psi_{A, X, Y, Z}(f)(a, [sx, (1-s)z]) = [s \cdot f(a, x), (1-s)z].$$

The remainder of this section is devoted to the definition and study of two maps $\alpha, \beta: A \times YZ \times I \rightarrow AYZ$, the second is simply the quotient onto $A(YZ) \cong AYZ$. They arise in the computations in Section 6 in the context of the homeomorphisms (4.5), and the next definition is our starting point. Recall that Δ^2 denotes the standard 2-simplex in \mathbb{R}^3 . For $i = 0, 1, 2$ let $\partial_i \Delta^2$ denote the i th face of Δ^2 , i.e the elements $(t_0, t_1, t_2) \in \Delta^2$ with $t_i = 0$.

Definition 4.3. Let $\tilde{\alpha}, \tilde{\beta}: I \times I \rightarrow \Delta^2$ be the functions

$$\begin{aligned}\tilde{\alpha}(s, t) &= (st, s(1-t), 1-s) \\ \tilde{\beta}(s, t) &= (t, s(1-t), (1-s)(1-t)).\end{aligned}$$

Both maps are clearly surjective, so for any $0 \leq s, t, \leq 1$ there exist $0 \leq s', t' \leq 1$ such that $\tilde{\beta}(s, t) = \tilde{\alpha}(s', t')$.

Proposition 4.4. Define functions $s', t': I \times I \rightarrow I$ as follows.

$$\begin{aligned}s'(s, t) &= s + t - st \\ t'(s, t) &= \begin{cases} 0 & \text{if } s = t = 0 \\ \frac{t}{s+t-st} & \text{if } (s, t) \neq (0, 0). \end{cases}\end{aligned}$$

- (a) s' is a continuous function, and t' is continuous away from $(0, 0)$.
- (b) $0 \leq s', t' \leq 1$
- (c) $\tilde{\beta} = \tilde{\alpha} \circ (s', t')$.
- (d) $s'(0, t) = t$ and $t'(s, 0) = 0$ and $t'(s, 1) = 1$. Also $t'(0, t) = 1$ for all $t \neq 0$.

Proof. First, $s + t - st = 1 - (1-s)(1-t)$. This shows that $0 \leq s' \leq 1$ and that the denominator in the formula for t' vanishes if and only if $s = t = 0$. This shows that t' is well defined and that it is continuous away from $(0, 0)$. The continuity of s' is clear. Also, $s + t - st = t + s(1-t) \geq t$ which shows that $0 \leq t' \leq 1$. This proves items (a) and (b). Items (c) and (d) follow by inspection of the formulas. \square

Of course, the maps $\tilde{\alpha}$ and $\tilde{\beta}$ are homotopic for trivial reasons. But we will need an explicit homotopy satisfying some conditions. Given a homotopy $h: I \times X \rightarrow Y$ we will write $h_p: X \rightarrow Y$ for the restriction of h to $\{p\} \times X$ where $0 \leq p \leq 1$.

Proposition 4.5. There exists a homotopy $\tilde{\theta}: I^2 \times I \rightarrow \Delta^2$ from $\tilde{\alpha}$ to $\tilde{\beta}$, written as a family of maps $\tilde{\theta}_p: I^2 \rightarrow \Delta^2$ parametrized by $0 \leq p \leq 1$, with the properties

$$\begin{aligned}\tilde{\theta}_p(\{0\} \times I) &\subseteq \partial_1 \Delta^2 & \tilde{\theta}_p(\{1\} \times I) &\subseteq \partial_2 \Delta^2 \\ \tilde{\theta}_p(I \times \{0\}) &\subseteq \partial_0 \Delta^2 & \tilde{\theta}_p(I \times \{1\}) &\subseteq \partial_1 \Delta^2\end{aligned}$$

Proof. Define functions $S, T: I \times I^2 \rightarrow I$ by

$$\begin{aligned}T(p, s, t) &= p \cdot t'(s, t) + (1-p)t, \\ S(p, s, t) &= p \cdot s'(s, t) + (1-p)s.\end{aligned}$$

It is clear from Proposition 4.4(b) and (a) that $0 \leq S, T \leq 1$ and that S is continuous and T is continuous away from $I \times \{(0, 0)\}$. Define functions $H, K: I \times I^2 \rightarrow \Delta^2$ as follows, where we write S, T instead of $S(p, s, t)$ and $T(p, s, t)$, and t' instead of $t'(s, t)$

$$\begin{aligned}H(p, s, t) &= (s \cdot T, s \cdot (1-T), 1-s) \\ K(p, s, t) &= (S \cdot t', S \cdot (1-t'), 1-S).\end{aligned}$$

They are well defined since by Propositions 4.4(b) $0 \leq t' \leq 1$ and we have seen that $0 \leq S, T \leq 1$. They are continuous away from $I \times \{(0, 0)\}$ because S, T are and t' is continuous away from $(0, 0)$. Also, it follows from Proposition 4.4(d) that

$$S_p(0, t) = pt \text{ and } S_p(1, t) = 1 \text{ and } T_p(s, 0) = 0 \text{ and } T_p(s, 1) = 1.$$

One then checks that

$$\begin{aligned}H_p(\{0\} \times I) &\subseteq \partial_1 \Delta^2 & K_p(\{0\} \times I) &\subseteq \partial_1 \Delta^2 \\ H_p(\{1\} \times I) &\subseteq \partial_2 \Delta^2 & K_p(\{1\} \times I) &\subseteq \partial_2 \Delta^2 \\ H_p(I \times \{0\}) &\subseteq \partial_0 \Delta^2 & K_p(I \times \{0\}) &\subseteq \partial_0 \Delta^2 \\ H_p(I \times \{1\}) &\subseteq \partial_1 \Delta^2 & K_p(I \times \{1\}) &\subseteq \partial_1 \Delta^2.\end{aligned}$$

Inspection of the definition of H and K gives

$$\begin{aligned} H_0(s, t) &= (st, s(1-t), 1-s) = \tilde{\alpha}(s, t) \\ H_1(s, t) &= (st', s(1-t'), 1-s) \\ K_0(s, t) &= (st', s(1-t'), 1-s) = H_1(s, t) \\ K_1(s, t) &= (s', t', s'(1-t'), 1-s') = (\tilde{\alpha} \circ (s', t'))(s, t) = \tilde{\beta}(s, t). \end{aligned}$$

Therefore the homotopies H_p and K_p can be concatenated to form a homotopy $\tilde{\Theta}: I \times I^2 \rightarrow \Delta^2$ from $\tilde{\alpha}$ to $\tilde{\beta}$ with the properties in the statement of this proposition. \square

Definition 4.6. Let A, Y, Z be compact Hausdorff spaces and assume that Y, Z are not empty. Let

$$\begin{aligned} q_{A,Y,Z}: A \times Y \times Z \times \Delta^2 &\rightarrow AYZ & \text{and} \\ q_{Y,Z}: Y \times Z \times I &\rightarrow YZ \end{aligned}$$

be the restriction of the quotient maps (4.1). Since $Y, Z \neq \emptyset$ the second map is a quotient map. Compactness of all spaces implies that $A \times q_{Y,Z} \times I$ in the left vertical map in the diagram below is a quotient map too. It can be described explicitly by the formula

$$(a, y, z, s, t) \mapsto (a, [sy, (1-s)z], t).$$

By Proposition 4.5 and inspection of the formula above, for any $0 \leq p \leq 1$ the composition of the top horizontal arrow with the vertical arrow on the right respects the quotient map on the left. We finally define $\Theta: (A \times YZ \times I) \times I \rightarrow AYZ$ to be the homotopy whose fibres Θ_p are the unique maps which render the following diagram commutative.

$$\begin{array}{ccc} A \times Y \times Z \times I \times I & \xrightarrow{A \times Y \times Z \times \tilde{\Theta}_p} & A \times Y \times Z \times \Delta^2 \\ \downarrow A \times q_{Y,Z} \times I & & \downarrow q_{A,Y,Z} \\ A \times YZ \times I & \dashrightarrow_{\Theta_p} & AYZ \end{array}$$

Definition 4.7. Let $\alpha, \beta: A \times YZ \times I \rightarrow AYZ$ be the maps $\alpha = \theta|_{p=0}$ and $\beta = \theta|_{p=1}$.

Proposition 4.8. *The maps α and β are homotopic and have the explicit formula*

$$\begin{aligned} \alpha(a, [sy, (1-s)z], t) &= [sta, s(1-t)y, (1-s)z] \\ \beta(a, [sy, (1-s)z], t) &= [ta, s(1-t)y, (1-s)(1-t)z] \end{aligned}$$

Proof. The homotopy is provided by θ_p . The formulas are immediate from the explicit description of $\tilde{\alpha}$ and $\tilde{\beta}$ in Definition 4.3 and Proposition 4.5. \square

The explicit formulas for α, β in Proposition 4.8 give the next straightforward calculation.

Proposition 4.9. (1) *The restriction $\alpha|_{A \times YZ \times \{0\}}: A \times YZ \rightarrow AYZ$ is the composition*

$$A \times YZ \xrightarrow{\text{proj}} YZ \xrightarrow{\text{incl}} AYZ.$$

(2) *The restriction $\alpha|_{A \times YZ \times \{1\}}: A \times YZ \rightarrow AYZ$ factors through the inclusion $AZ \subseteq AYZ$ and is given by $(a, [sy, (1-s)z]) \mapsto [sa, (1-s)z]$.*

(3) *The restriction $\beta|_{A \times YZ \times \{1\}}: A \times YZ \rightarrow AYZ$ is the composition $A \times YZ \xrightarrow{\text{proj}} A \xrightarrow{\text{incl}} AYZ$.*

The following facts are again straightforward calculations:

Proposition 4.10. *Let A, X, Y be compact Hausdorff G -spaces, $X, Y \neq \emptyset$. Recall the maps J_Z and ψ from (4.3) and Definition 4.2, and let $\pi: A \times Y \times I \rightarrow AY$ be the restriction of the*

quotient map (4.1). Then the following diagrams commute.

$$(4.8) \quad \begin{array}{ccccc} \text{map}^G(AY, XY) & \xrightarrow{\pi^*} & \text{map}^G(A \times Y \times I, XY) & \xrightarrow{\cong} & \text{map}^G(A \times I \times Y, XY) \\ \downarrow J_Z & & & & \downarrow \psi_{A \times I, Y, XY, Z} \\ \text{map}^G(AYZ, XYZ) & \xrightarrow{\alpha^*} & \text{map}^G(A \times YZ \times I, XYZ) & \xrightarrow{\cong} & \text{map}^G(A \times I \times YZ, XYZ) \end{array}$$

$$(4.9) \quad \begin{array}{ccc} \text{map}^G(Y, XY) & \xrightarrow{J_Z} & \text{map}^G(YZ, XYZ) \\ \pi^* \downarrow & & \downarrow (\alpha|_{A \times YZ \times \{0\}})^* \\ \text{map}^G(A \times Y, XY) & \xrightarrow{\psi_{A, Y, XY, Z}} & \text{map}^G(A \times YZ, XYZ) \end{array}$$

$$(4.10) \quad \begin{array}{ccc} \text{map}^G(A, XY) & \xrightarrow{J_Z} & \text{map}^G(AZ, XYZ) \\ \pi^* \downarrow & & \downarrow (\alpha|_{A \times YZ \times \{1\}})^* \\ \text{map}^G(A \times Y, XY) & \xrightarrow{\psi_{A, Y, XY, Z}} & \text{map}^G(A \times YZ, XYZ) \end{array}$$

$$(4.11) \quad \begin{array}{ccc} \text{map}^G(A, XY) & \xrightarrow{J_Z} & \text{map}^G(AZ, XYZ) \\ \text{incl}_* \downarrow & \swarrow i^* & \downarrow (\beta|_{A \times YZ \times \{1\}})^* \\ \text{map}^G(A, XY) & \xrightarrow{\pi^*} & \text{map}^G(A \times YZ, XYZ) \end{array}$$

Proof. To check (4.8) we use the formula for α in Proposition 4.8, for J_Z in (4.4) and for ψ in (4.7), to calculate

$$\begin{aligned} \alpha^* J_Z(f)(a, t, [sy, (1-s)z]) &= J_Z(f)([st \cdot a, s(1-t) \cdot y, (1-s) \cdot z]) = [s \cdot f([ta, (1-t)y], (1-s)z)] = \\ &= [s \cdot (\pi^* f)(a, t, y), (1-s)z] = \psi_{A \times I, Y, XY, Z}(\pi^* f)(a, t, [sy, (1-s)z]). \end{aligned}$$

The commutativity of (4.9) follows from Proposition 4.9(1), the naturality of ψ with respect to $A \rightarrow *$, and the observation that J_Z is $\psi_{*, Y, XY, Z}$.

The commutativity of (4.10) follows by the following calculation which uses Proposition 4.9(2), and equations (4.4) and (4.7).

$$\begin{aligned} (\alpha_1^* J_Z(h))(a, t, [sy, (1-s)z]) &= J_Z(h)([sa, [1-s]z]) = \\ &= [s \cdot h(a), (1-s)z] = [s \cdot (\pi^* h)(a, y), (1-s)z] = \psi_{A, Y, XY, Z}(\pi^* h)(a, [sy, (1-s)z]). \end{aligned}$$

Finally, (4.11) follows from (4.6) and Proposition 4.9(3). \square

5. FILTRATION OF G -SPACES

Let G be a finite group. Let (H) denote the conjugacy class of $H \leq G$. Enumerate the conjugacy classes of G

$$(5.1) \quad (H_1), \dots, (H_r)$$

so that $|H_i| \geq |H_{i+1}|$. In this way, if H_i is conjugate to a proper subgroup of H_j then $i > j$.

Let X be a G -space. Let G_x denote the isotropy group of $x \in X$. For any $0 \leq q \leq r$ set

$$(5.2) \quad X_q = \{x \in X : G_x \in (H_i) \text{ for some } 1 \leq i \leq q\}.$$

We obtain a filtration of X

$$\emptyset = X_0 \subseteq X_1 \subseteq \dots \subseteq X_r = X.$$

See [11, Chap. I.6] If X is a G -CW complex then one checks that X_q are subcomplexes [11, Prop. II.1.12]. The assignment $X \mapsto X_q$ is a functor giving rise to natural maps

$$\text{map}^G(X, Y) \xrightarrow{f \mapsto f|_{X_q}} \text{map}^G(X_q, Y_q).$$

This is because $f(X_q) \subseteq Y_q$ by the choice of the enumeration (5.1), see [11, I.(6.3)].

Proposition 5.1. *Let X, Y be compact Hausdorff G -spaces.*

- (1) *If $H \leq G$ then $(XY)^H = X^H Y^H$ (where XY denotes the join).*
(2) *For any $1 \leq q \leq r$*

$$(XY)_q \subseteq X_q Y_q \subseteq X_q Y \subseteq XY$$

- (3) *Set $H = H_q$ for some $1 \leq q \leq r$. Then*

$$\begin{aligned} X_q^H &= X^H \\ X_{q-1}^H &= \bigcup_{K \geq H} X^K. \end{aligned}$$

Proof. Notice that the isotropy group of $[x, y, t] \in XY$ is G_x if $t = 1$ and G_y if $t = 0$ and $G_x \cap G_y$ if $0 < t < 1$. From this items (1) and (2) follow easily.

Set $H = H_q$ for some $1 \leq q \leq r$. By construction

$$X_q \setminus X_{q-1} = \{x \in X : G_x \in (H)\}.$$

The choice of the enumeration (5.1) implies item (3) since $x \in X^H$ if and only if $H \leq G_x$. \square

Recall that if X is a G -space and $H \leq G$ then X^H admits an action of $WH = N_G H / H$. This gives a functor $X \mapsto X^H$ from G -spaces to WH -spaces. There results a natural map

$$(5.3) \quad \text{map}^G(X, Y) \xrightarrow{\text{res}_G^H: f \mapsto f|_{X^H}} \text{map}^{WH}(X^H, Y^H)$$

Proposition 5.2. *Let X, Y, Z be compact Hausdorff G -spaces.*

- (1) *The join map J_Z (4.3) renders the following square commutative*

$$\begin{array}{ccc} \text{map}^G(X, Y) & \xrightarrow{\text{res}_G^H} & \text{map}^{WH}(X^H, Y^H) \\ J_Z \downarrow & & \downarrow J_{ZH} \\ \text{map}^G(XZ, YZ) & \xrightarrow{\text{res}_G^H} & \text{map}^{WH}((XZ)^H, (YZ)^H). \end{array}$$

- (2) *Let $H \leq G$. Then*

$$\text{Iso}_{WH}(X^H, X_{q-1}^H) \subseteq \{e\}.$$

Proof. Item (1) follows from Proposition 5.1(1) and by inspection of (4.4). For item (2), suppose that $x \in X^H \setminus X_{q-1}^H$. Then $H \leq G_x$ and by choice of the enumeration (5.1), $G_x \in (H_i)$ for some $i \leq q$. Since $x \notin X_{q-1}$ it follows that $i = q$ and therefore $G_x = H$. In particular, WH_x is trivial. \square

Proposition 5.3. *Let X be a G -CW complex. Set $H = H_q \in \text{Iso}_G(X)$ for some $1 \leq q \leq r$. Then there is a pullback square, natural in both X and Y*

$$\begin{array}{ccc} \text{map}^G(X_q, Y) & \xrightarrow{i^*} & \text{map}^G(X_{q-1}, Y) \\ \text{res}_G^H \downarrow & & \downarrow \text{res}_G^H \\ \text{map}^{WH}(X^H, Y^H) & \xrightarrow{i^*} & \text{map}^{WH}(X_{q-1}^H, Y^H) \end{array}$$

whose rows are fibrations, hence the vertical arrows induce homeomorphisms on all fibres.

Proof. Since $\text{Iso}_G(X_q, X_{q-1})$ is the conjugacy class of $H = H_q$, there is a pushout square

$$(5.4) \quad \begin{array}{ccc} G \times_{NH} X_{q-1}^H & \xrightarrow{(g,x) \mapsto gx} & X_{q-1} \\ \downarrow & & \downarrow \\ G \times_{NH} X_q^H & \xrightarrow{(g,x) \mapsto gx} & X_q \end{array}$$

in which the vertical arrows are inclusions of G -CW complexes, i.e G -cofibrations, and which is natural in X . The pullback square is obtained upon applying the functor $\text{map}^G(-, Y)$ to this pushout square and observing that $X_q^H = X^H$ (Proposition 5.1(3)) and that if A is a WH -space then there are natural homeomorphisms

$$\text{map}^G(G \times_{NH} A, Y) \cong \text{map}^{NH}(A, Y) = \text{map}^{NH}(A, Y^H) = \text{map}^{WH}(A, Y^H).$$

□

6. THE STABILIZATION LEMMA

The purpose of this section is to prove the following Proposition.

Proposition 6.1. *Let X, Y, Z be finite G -CW complexes. Let $k \geq 0$. Assume that*

$$(1) \text{Iso}_G(X) = \text{Iso}_G(Y) = \text{Iso}_G(XYZ),$$

and that for any $H \in \text{Iso}_G(X)$

$$(2) \dim Y^H > k,$$

$$(3) \dim X^H - \dim \left(\bigcup_{K \geq H} X^K \right) > k.$$

$$(4) \text{conn}(XY)^H \geq \dim X^H + \dim Y^H \text{ and } \text{conn}(XYZ)^H \geq \dim X^H + \dim(YZ)^H$$

$$(5) F(Y^H, (XY)^H) \xrightarrow{J_Z^H} F((YZ)^H, (XYZ)^H) \text{ is a non-equivariant } (\dim X^H + k + 1)\text{-equivalence (see (4.3) and Proposition 5.1(1)).}$$

Then the natural map

$$\text{map}^G(XY, XY) \xrightarrow{J_Z} \text{map}^G(XYZ, XYZ)$$

is a k -equivalence.

Proof. We will use the filtration (5.2) and show that the composition

$$(6.1) \quad \text{map}^G((XY)_q, XY) \xrightarrow{J_Z} \text{map}^G((XY)_q Z, XYZ) \xrightarrow{j^*} \text{map}^G((XYZ)_q, XYZ)$$

is a k -equivalence for any $0 \leq q \leq r$, where j denotes the inclusion $(XYZ)_q \subseteq (XY)_q Z$, see Proposition 5.1(2). The claim of the proposition follows for $q = r$.

The proof is by induction on $0 \leq q \leq r$. The base of induction $q = 0$ is a triviality since $(XY)_q = (XYZ)_q = \emptyset$. We therefore assume that (6.1) is a k -equivalence for $q - 1$ and we prove it for $q \leq r$. Set

$$H = H_q.$$

Since $X \subseteq XY \subseteq XYZ$, it follows from hypothesis (1) that $\text{Iso}_G(X) = \text{Iso}_G(XY) = \text{Iso}_G(XYZ)$. By the definition of the filtration (5.2), if $H \notin \text{Iso}_G(X)$ then $(XY)_q = (XY)_{q-1}$ and $(XYZ)_q = (XYZ)_{q-1}$, in which case the induction step follows trivially from its hypothesis. We therefore assume that

$$H \in \text{Iso}_G(X).$$

Naturality of J_Z gives the following commutative diagram whose rows are fibrations and in which i denotes the inclusions $(XY)_{q-1} \subseteq (XY)_q$ and $(XYZ)_{q-1} \subseteq (XYZ)_q$, and j denotes

the inclusions $(XYZ)_q \subseteq (XY)_q Z$ and $(XYZ)_{q-1} \subseteq (XY)_{q-1} Z$.

$$(6.2) \quad \begin{array}{ccc} \text{map}^G((XY)_q, XY) & \xrightarrow{i^*} & \text{map}^G((XY)_{q-1}, XY) \\ j^* \circ J_Z \downarrow & & \downarrow j^* \circ J_Z \\ \text{map}^G((XYZ)_q, XYZ) & \xrightarrow{i^*} & \text{map}^G((XYZ)_{q-1}, XYZ). \end{array}$$

The vertical arrow on the right is a k -equivalence by the induction hypothesis, so by Lemma 2.2(1) it remains to show that the fibres of i^* are k -equivalent.

Proposition 5.1(1) and the formula (4.4) for $J_Z(f)$ easily imply the commutativity of the following diagram, where i and ℓ denote the inclusions of the H -fixed points of $(XY)_{q-1} \subseteq (X_{q-1}Y)_q \subseteq (XY)_q$ and $(XYZ)_{q-1} \subseteq (X_{q-1}YZ)_q \subseteq (XYZ)_q$, and j denoted the inclusion of the H -fixed points of $(XYZ)_{q-1} \subseteq (XY)_{q-1} Z$.

$$(6.3) \quad \begin{array}{ccc} \text{map}^{WH}((XY)^H, (XY)^H) & \xrightarrow{i^*} & \text{map}^{WH}((X_{q-1}Y)^H, (XY)^H) \\ \parallel & & \downarrow \ell^* \\ \text{map}^{WH}((XY)^H, (XY)^H) & \xrightarrow{i^*} & \text{map}^{WH}((XY)_{q-1}^H, (XY)^H) \\ J_{ZH} \downarrow & & \downarrow j^* \circ J_{ZH} \\ \text{map}^{WH}((XYZ)^H, (XYZ)^H) & \xrightarrow{i^*} & \text{map}^{WH}((XYZ)_{q-1}^H, (XYZ)^H) \\ \parallel & & \uparrow \ell^* \\ \text{map}^{WH}((XYZ)^H, (XYZ)^H) & \xrightarrow{i^*} & \text{map}^{WH}((X_{q-1}YZ)^H, (XYZ)^H) \end{array}$$

By Proposition 5.2(1), the maps res_G^H in (5.3) give rise to a natural transformation between the commutative square (6.2) to the square in the middle of (6.3). By Proposition 5.3 the fibres of the rows of (6.2) over any $f \in \text{map}^G((XY)_{q-1}, XY)$ are homeomorphic to the fibres over $\text{res}_G^H(f)$ of the rows of the middle square of (6.3).

Therefore, it suffices to prove that the fibres of the rows of the 2nd square in (6.3) are k -equivalent for any choice of basepoint in $\text{map}^{WH}((XY)_{q-1}^H, (XY)^H)$. If $Z^H = \emptyset$ then this is a triviality since J_{ZH} and j^* are the identity maps. So for the remainder of the proof we assume that

$$Z^H \neq \emptyset.$$

It follows from hypothesis (3) and from Proposition 5.1(3) that $\dim X^H - \dim X_{q-1}^H \geq k + 1$. Together with hypotheses (4) and (2) we get

$$\text{conn}(XY)^H - \dim(X_{q-1}Y)^H \geq \dim X^H + \dim Y^H - (\dim X_{q-1}^H + \dim Y^H + 1) \geq k.$$

Similarly,

$$\text{conn}(XYZ)^H \geq \dim(X_{q-1}YZ)^H + k.$$

Since $H \in \text{Iso}_G(X)$ Proposition 5.2(2) implies that $\text{Iso}_{WH}((X_{q-1}Y)^H, (XY)_{q-1}^H) = \{e\}$. Corollary 2.5 applies with $(XY)_{q-1}^H \subseteq (X_{q-1}Y)^H$ and with $(XYZ)_{q-1}^H \subseteq (X_{q-1}YZ)^H$ to show that both maps ℓ^* in (6.3) are k -equivalences. It follows from Lemma 2.2(2) that the fibres of i^* at the top and bottom squares of (6.3) are k -equivalent. Since ℓ^* are bijective on components, it suffices to show that the fibres of i^* at the top and bottom of (6.3) are k -equivalent via the curved arrows.

This is indeed the case by applying Proposition 6.2 below with $X_{q-1}^H \subseteq X^H$ and Y^H and Z^H and $G = WH$. To see this, notice first that X^H and Y^H are not empty since $H \in \text{Iso}_G(X) = \text{Iso}_G(Y)$. Also $Z^H \neq \emptyset$ by assumption. Hypothesis (1) of Proposition 6.2 follows from Proposition 5.2(2). Hypothesis (2) of Proposition 6.2 is hypothesis (5) of this proposition. Hypothesis (3) of Proposition 6.2 follows from hypotheses (4) and (2) of this proposition. \square

Proposition 6.2. *Let G be a finite group and X, Y, Z be finite non-empty G -CW complexes and $X' \subseteq X$ a G -subcomplex. Let $k \geq 0$. Suppose that*

- (1) $\text{Iso}_G(X, X') = \{e\}$.
- (2) $F(Y, XY) \xrightarrow{J_Z} F(YZ, XYZ)$ is a non-equivariant $(\dim X + k + 1)$ -equivalence.
- (3) $\text{conn}(XY) \geq \dim X + k + 1$.

Then the maps induced on fibres of the horizontal arrows in the following diagram

$$\begin{array}{ccc} \text{map}^G(XY, XY) & \xrightarrow{i^*} & \text{map}^G(X'Y, XY) \\ J_Z \downarrow & & \downarrow J_Z \\ \text{map}^G(XYZ, XYZ) & \xrightarrow{i^*} & \text{map}^G(X'YZ, XYZ) \end{array}$$

are k -equivalences for any choice of basepoint in the space at the top right corner.

Proof. Let $i: X' \rightarrow X$ be the inclusion. Define the following objects in \mathfrak{S} , see Definition 3.1.

$$\begin{array}{l} \mathcal{A} = \begin{array}{ccc} \text{map}^G(XY, XY) & \xrightarrow{(i_Y^*, i_X^*)} & \text{map}^G(Y, XY) \times \text{map}^G(X, XY) \\ i^* \downarrow & & \downarrow \text{id} \times i^* \\ \text{map}^G(X'Y, XY) & \xrightarrow{(i_Y^*, i_{X'}^*)} & \text{map}^G(Y, XY) \times \text{map}^G(X', XY). \end{array} \\ \mathcal{B} = \begin{array}{ccc} \text{map}^G(XYZ, XYZ) & \xrightarrow{(i_{YZ}^*, i_X^*)} & \text{map}^G(YZ, XYZ) \times \text{map}^G(X, XYZ) \\ i^* \downarrow & & \downarrow \text{id} \times i^* \\ \text{map}^G(X'YZ, XYZ) & \xrightarrow{(i_{YZ}^*, i_{X'}^*)} & \text{map}^G(YZ, XYZ) \times \text{map}^G(X', XYZ). \end{array} \\ \mathcal{C} = \begin{array}{ccc} \text{map}^G(X \times Y \times I, XY) & \xrightarrow{(\text{ev}_0, \text{ev}_1)} & \text{map}^G(X \times Y, XY)^2 \\ i^* \downarrow & & \downarrow i^* \times i^* \\ \text{map}^G(X' \times Y \times I, XY) & \xrightarrow{(\text{ev}_0, \text{ev}_1)} & \text{map}^G(X' \times Y, XY)^2. \end{array} \\ \mathcal{D} = \begin{array}{ccc} \text{map}^G(X \times YZ \times I, XYZ) & \xrightarrow{(\text{ev}_0, \text{ev}_1)} & \text{map}^G(X \times YZ, XYZ)^2 \\ i^* \downarrow & & \downarrow i^* \times i^* \\ \text{map}^G(X' \times YZ \times I, XYZ) & \xrightarrow{(\text{ev}_0, \text{ev}_1)} & \text{map}^G(X' \times YZ, XYZ)^2 \end{array} \end{array}$$

The commutativity of these squares is a direct consequence of the naturality of $i \mapsto i^*$. The plan of the proof is as follows.

- (a) Define morphisms $\mathcal{A} \xrightarrow{\Phi} \mathcal{B} \xrightarrow{\Pi} \mathcal{D}$ and $\mathcal{A} \xrightarrow{\Pi} \mathcal{C} \xrightarrow{\Psi} \mathcal{D}$ in \mathfrak{S} . *Note:* We used Π to denote two different morphisms; This will create no source of confusion and the reason for the choice will become apparent in (6.5) and (6.6) where they are defined.
- (b) Show that both $\Lambda(\mathcal{B}, \underline{y}) \xrightarrow{\Lambda(\Pi)} \Lambda(\mathcal{D}, \Pi(\underline{y}))$ and $\Lambda(\mathcal{A}, \underline{x}) \xrightarrow{\Lambda(\Pi)} \Lambda(\mathcal{C}, \Pi(\underline{x}))$ are weak homotopy equivalences for any choice of basepoints \underline{x} for \mathcal{A} and \underline{y} for \mathcal{B} , see Definition 3.1 and equation (3.2).
- (c) Show that $\Lambda(\mathcal{C}, \underline{y}) \xrightarrow{\Lambda(\Psi)} \Lambda(\mathcal{D}, \Psi(\underline{y}))$ is a k -equivalence for any choice of basepoint \underline{y} in \mathcal{C} .
- (d) Show that $\mathcal{A} \xrightarrow{\Pi \circ \Phi} \mathcal{D}$ is homotopic to $\mathcal{A} \xrightarrow{\Psi \circ \Pi} \mathcal{D}$ (Definition 3.3).
- (e) Deduce that $\Lambda(\mathcal{A}, \underline{x}) \xrightarrow{\Lambda(\Phi)} \Lambda(\mathcal{B}, \Phi(\underline{x}))$ is a k -equivalence for any choice of base point \underline{x} for \mathcal{A} . Use this and Lemma 3.2(3) to complete the proof.

With the indexing in (3.1), we will now describe maps $\Phi_i: \mathcal{A}_i \rightarrow \mathcal{B}_i$ and $\Pi_i: \mathcal{B}_i \rightarrow \mathcal{D}_i$ and $\Pi_i: \mathcal{A}_i \rightarrow \mathcal{C}_i$ and $\Psi_i: \mathcal{C}_i \rightarrow \mathcal{D}_i$, where $i = 0, \dots, 3$. We will then show that these are the components of natural transformations $\mathcal{A} \xrightarrow{\Phi} \mathcal{B}$ and $\mathcal{B} \xrightarrow{\Pi} \mathcal{D}$ and $\mathcal{A} \xrightarrow{\Pi} \mathcal{C}$ and $\mathcal{C} \xrightarrow{\Psi} \mathcal{D}$.

Notation: In what follows we will use the letter A to represents either X or X' ,

$$A = X' \text{ or } A = X.$$

Define $\Phi_i: \mathcal{A}_i \rightarrow \mathcal{B}_i$ using the maps J_Z (4.3) and the inclusion $\text{incl}: XY \rightarrow XYZ$ as follows,

$$(6.4) \quad \begin{aligned} \Phi_1, \Phi_3: \text{map}^G(AY, XY) &\xrightarrow{J_Z} \text{map}^G(AYZ, XYZ) \\ \Phi_0, \Phi_2: \text{map}^G(Y, XY) \times \text{map}^G(A, XY) &\xrightarrow{J_Z \times \text{incl}_*} \text{map}^G(YZ, XYZ) \times \text{map}^G(A, XYZ) \end{aligned}$$

Let $\pi: A \times YZ \times I \rightarrow AYZ$ be the map in (4.2). Let $\pi_A: A \times YZ \rightarrow A$ and $\pi_{YZ}: A \times YZ \rightarrow YZ$ be the projections. Define $\Pi_i: \mathcal{B}_i \rightarrow \mathcal{D}_i$ as follows,

$$(6.5) \quad \begin{aligned} \Pi_1, \Pi_3: \text{map}^G(AYZ, XYZ) &\xrightarrow{\pi^*} \text{map}^G(A \times YZ \times I, XYZ) \\ \Pi_0, \Pi_2: \text{map}^G(YZ, XYZ) \times \text{map}^G(A, XYZ) &\xrightarrow{\pi_{YZ}^* \times \pi_A^*} \text{map}^G(A \times YZ, XYZ)^2 \end{aligned}$$

Let $\pi: A \times Y \times I \rightarrow AY$ be as in (4.2) and $\pi_A: A \times Y \rightarrow A$ and $\pi_Y: A \times Y \rightarrow Y$ be the projections. Set $\Pi_i: \mathcal{A}_i \rightarrow \mathcal{C}_i$ as follows

$$(6.6) \quad \begin{aligned} \Pi_3, \Pi_1: \text{map}^G(AY, XY) &\xrightarrow{\pi^*} \text{map}^G(A \times Y \times I, XY) \\ \Pi_2, \Pi_0: \text{map}^G(Y, XY) \times \text{map}^G(A, XY) &\xrightarrow{\pi_Y^* \times \pi_A^*} \text{map}^G(A \times Y, XY)^2 \end{aligned}$$

Use the maps $\psi_{A \times I, Y, XYZ}$ and $\psi_{A, Y, XYZ}$ in Definition 4.2 to define $\Psi_i: \mathcal{C}_i \rightarrow \mathcal{D}_i$

$$(6.7) \quad \begin{aligned} \Psi_3, \Psi_1: \text{map}^G(A \times Y \times I, XY) &\xrightarrow{\psi} \text{map}^G(A \times YZ \times I, XYZ) \\ \Psi_2, \Psi_0: \text{map}^G(A \times Y, XY)^2 &\xrightarrow{\psi \times \psi} \text{map}^G(A \times YZ, XYZ)^2 \end{aligned}$$

Claim 1: The maps Φ_0, \dots, Φ_3 in (6.4) define a natural transformation $\Phi: \mathcal{A} \rightarrow \mathcal{B}$.

Proof: With the indexing of Definition 3.1, the naturality of J_Z and the equality $\text{incl}_* \circ i^* = i^* \circ \text{incl}_*$ imply that $\Phi_1 \circ a_{31} = b_{31} \circ \Phi_3$ and $\Phi_0 \circ a_{20} = b_{20} \circ \Phi_2$. The commutative square in (4.6) implies the commutativity of the following diagram

$$\begin{array}{ccc} \text{map}^G(AY, XY) &\xrightarrow{((i_Y^{AY})^*, (i_A^{AY})^*)} & \text{map}^G(Y, XY) \times \text{map}^G(A, XY) \\ J_Z \downarrow & & \downarrow J_Z \times J_Z \\ \text{map}^G(AYZ, XYZ) &\xrightarrow{((i_Y^{AYZ})^*, (i_A^{AYZ})^*)} & \text{map}^G(YZ, XYZ) \times \text{map}^G(AZ, XYZ) \end{array}$$

Composing the 2nd factor of the 2nd column with $\text{map}^G(AZ, XYZ) \xrightarrow{i_A^*} \text{map}^G(A, XYZ)$ and using the commutative triangle in (4.6), it follows that $\Phi_2 \circ a_{32} = b_{32} \circ \Phi_3$ and $\Phi_0 \circ a_{10} = b_{10} \circ \Phi_1$. Hence, $\Phi: \mathcal{A} \rightarrow \mathcal{B}$ is a morphism in \mathfrak{S} . QED

Claim 2: The maps Ψ_0, \dots, Ψ_3 in (6.7) define a morphism $\Psi: \mathcal{C} \rightarrow \mathcal{D}$.

Proof: This is immediate from the naturality of ψ with respect to the inclusions $X' \subseteq X$ and $X' \times I \subseteq X \times I$ and the inclusions $A \amalg A \subseteq A \times I$. QED.

Claim 3: The maps Π_0, \dots, Π_3 in (6.6) define a morphism $\Pi: \mathcal{A} \rightarrow \mathcal{C}$ in \mathfrak{S} . Moreover $\Lambda(\mathcal{A}, \underline{x}) \xrightarrow{\Lambda(\Pi)} \Lambda(\mathcal{C}, \Pi(\underline{x}))$ is a weak homotopy equivalence for any basepoint \underline{x} for \mathcal{A} . Similarly, the maps Π_0, \dots, Π_3 in (6.5) define a morphism $\Pi: \mathcal{B} \rightarrow \mathcal{D}$ in \mathfrak{S} and $\Lambda(\mathcal{B}, \underline{y}) \xrightarrow{\Lambda(\Pi)} \Lambda(\mathcal{D}, \Pi(\underline{y}))$ is a weak homotopy equivalence for any basepoint \underline{y} in \mathcal{B} .

Proof: We will prove the statements about the maps in (6.6) and $\Pi: \mathcal{A} \rightarrow \mathcal{C}$. The proof for the maps in (6.5) and $\Pi: \mathcal{B} \rightarrow \mathcal{D}$ is obtained by replacing Y with YZ everywhere and \mathcal{A} with \mathcal{B} and \mathcal{C} with \mathcal{D} .

By applying $\text{map}^G(-, XY)$ to the commutative squares

$$\begin{array}{ccc} X' \times Y \times I & \xrightarrow{\pi} & X'Y \\ \downarrow i & & \downarrow i \\ X \times Y \times I & \xrightarrow{\pi} & XY \end{array} \quad \begin{array}{ccc} (X' \times Y) \amalg (X' \times Y) & \xrightarrow{\pi_Y \amalg \pi_{X'}} & Y \amalg X' \\ \downarrow & & \downarrow \\ (X \times Y) \amalg (X \times Y) & \xrightarrow{\pi_Y \amalg \pi_X} & Y \amalg X. \end{array}$$

it follows that $c_{31} \circ \Pi_3 = \Pi_1 \circ a_{31}$ and $c_{20} \circ \Pi_2 = \Pi_0 \circ a_{20}$. By applying $\text{map}^G(-, XY)$ to the pushout square (4.2) we obtain pullback squares

$$\begin{array}{ccc} \mathcal{A}_3 & \xrightarrow{a_{32}} & \mathcal{A}_2 \\ \Pi_3 \downarrow & & \downarrow \Pi_2 \\ \mathcal{C}_3 & \xrightarrow{c_{32}} & \mathcal{C}_2 \end{array} \quad \begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{a_{10}} & \mathcal{A}_0 \\ \Pi_1 \downarrow & & \downarrow \Pi_0 \\ \mathcal{C}_1 & \xrightarrow{c_{10}} & \mathcal{C}_0 \end{array}$$

which are in particular commutative. Thus, Π_0, \dots, Π_3 define a natural transformation $\Pi: \mathcal{A} \rightarrow \mathcal{C}$. Since the horizontal arrows a_{32}, a_{10} and c_{32} and c_{10} are fibrations, the two squares above are homotopy pullback squares and Lemma 3.2(1) shows that $\Lambda(\Pi, \underline{x})$ are weak homotopy equivalences for all basepoints \underline{x} of \mathcal{A} . QED.

Claims 1–3 complete steps (a) and (b) in our plan of the proof.

Claim 4: $\Lambda(\mathcal{C}, \underline{y}) \xrightarrow{\Lambda(\Psi)} \Lambda(\mathcal{D}, \Psi(\underline{y}))$ is a k -equivalence for any choice of basepoint \underline{y} in \mathcal{C} .

Proof: Hypotheses (1) and (2) allow us to apply Lemma 2.4 to the following commutative squares

$$\begin{array}{ccc} \text{map}^G(X \times I, F(Y, XY)) & \xrightarrow{i^*} & \text{map}^G(X' \times I, F(Y, XY)) \\ J_Z^* \downarrow & & \downarrow J_Z^* \\ \text{map}^G(X \times I, F(YZ, XYZ)) & \xrightarrow{i^*} & \text{map}^G(X' \times I, F(YZ, XYZ)) \end{array}$$

$$\begin{array}{ccc} \text{map}^G(X, F(Y, XY))^2 & \xrightarrow{i^* \times i^*} & \text{map}^G(X', F(Y, XY))^2 \\ J_Z^* \times J_Z^* \downarrow & & \downarrow J_Z^* \times J_Z^* \\ \text{map}^G(X, F(YZ, XYZ))^2 & \xrightarrow{i^* \times i^*} & \text{map}^G(X', F(YZ, XYZ))^2 \end{array}$$

It follows that the fibres of the rows in the 1st square are k -equivalent and those in the 2nd square are $(k+1)$ -equivalent. By construction of the maps Ψ_0, \dots, Ψ_3 in (6.7) and Definition 4.2 of the maps ψ , it follows that the maps induced on the homotopy fibres in the following diagrams

$$\begin{array}{ccc} \text{hoFib}(c_{31}, y_1) & \longrightarrow & \mathcal{C}_3 \xrightarrow{c_{31}} \mathcal{C}_1 \\ \downarrow & & \Psi_3 \downarrow \quad \downarrow \Psi_1 \\ \text{hoFib}(d_{31}, \Psi_1(y_1)) & \longrightarrow & \mathcal{D}_3 \xrightarrow{d_{31}} \mathcal{D}_1 \end{array} \quad \begin{array}{ccc} \text{hoFib}(c_{20}, y_0) & \longrightarrow & \mathcal{C}_2 \xrightarrow{c_{20}} \mathcal{C}_0 \\ \downarrow & & \Psi_2 \downarrow \quad \downarrow \Psi_0 \\ \text{hoFib}(d_{20}, \Psi_0(y_0)) & \longrightarrow & \mathcal{D}_2 \xrightarrow{d_{20}} \mathcal{D}_0 \end{array}$$

are $(k+1)$ -equivalences for any choice of basepoints $y_0 \in \mathcal{C}_0$ and $y_1 \in \mathcal{C}_1$. Lemma 3.2(2) shows that $\Lambda(\mathcal{C}, \underline{y}) \xrightarrow{\Lambda(\Psi)} \Lambda(\mathcal{D}, \Psi(\underline{y}))$ is a k -equivalence for any basepoint \underline{y} of \mathcal{C} . QED.

This completes step (c) of the proof. We turn to the technical proof of step (d).

Claim 5: The morphisms $\mathcal{A} \xrightarrow{\Phi} \mathcal{B} \xrightarrow{\Pi} \mathcal{D}$ and $\mathcal{A} \xrightarrow{\Pi} \mathcal{C} \xrightarrow{\Psi} \mathcal{D}$ are homotopic (Definition 3.3).

Proof: The plan is to define an object $\mathcal{H} \in \mathfrak{S}$, a morphism $\mathcal{A} \xrightarrow{\Upsilon} \mathcal{H}$ and a homotopy $\mathcal{H} \xrightarrow{\Xi_p} \mathcal{D}$ parametrized by $0 \leq p \leq 1$, such that $(\Xi \circ \Upsilon)|_{p=0} = \Pi \circ \Phi$ and $(\Xi \circ \Upsilon)|_{p=1} = \Psi \circ \Pi$.

By applying the functor $\text{map}^G(-, XYZ)$ to the commutative square of inclusions

$$\begin{array}{ccc} YZ \sqcup X'Z & \hookrightarrow & X'YZ \\ \downarrow & & \downarrow \\ YZ \sqcup XZ & \hookrightarrow & XYZ \end{array}$$

we obtain the following object \mathcal{H} in \mathfrak{S} .

$$\mathcal{H} = \begin{array}{ccc} \text{map}^G(XYZ, XYZ) & \xrightarrow{(i_{YZ}^*, i_{XZ}^*)} & \text{map}^G(YZ, XYZ) \times \text{map}^G(XZ, XYZ) \\ i^* \downarrow & & \downarrow \text{id} \times i^* \\ \text{map}^G(X'YZ, XYZ) & \xrightarrow{(i_{YZ}^*, i_{X'Z}^*)} & \text{map}^G(YZ, XYZ) \times \text{map}^G(X'Z, XYZ). \end{array}$$

Define maps $\Upsilon_i: \mathcal{A}_i \rightarrow \mathcal{H}_i$ ($i = 0, \dots, 3$) as follows (we use the indexing as in Definition 3.1).

$$(6.8) \quad \Upsilon_1, \Upsilon_3: \text{map}^G(AY, XY) \xrightarrow{J_Z} \text{map}^G(AYZ, XYZ)$$

$$\Upsilon_0, \Upsilon_2: \text{map}^G(Y, XY) \times \text{map}^G(A, XY) \xrightarrow{J_Z \times J_Z} \text{map}^G(YZ, XYZ) \times \text{map}^G(AZ, XYZ).$$

They give rise to a morphism $\Upsilon: \mathcal{A} \rightarrow \mathcal{H}$ in \mathfrak{S} by commutativity of the square in (4.6).

We now define homotopies $(\Xi_i)_p: \mathcal{H}_i \rightarrow \mathcal{D}_i$ parametrized by $0 \leq p \leq 1$. Apply $\text{map}^G(-, XYZ)$ to the homotopies $\Theta_p: A \times YZ \times I \rightarrow AYZ$ in Definition 4.6 to obtain the maps

$$(6.9) \quad (\Xi_3)_p: \mathcal{H}_3 \xrightarrow{(\Theta_p)^*} \mathcal{D}_3 \quad \text{and} \quad (\Xi_1)_p: \mathcal{H}_1 \xrightarrow{(\Theta_p)^*} \mathcal{D}_1.$$

Definition 4.6 and Proposition 4.5 show that $\Theta_p|_{t=0} \stackrel{\text{def}}{=} \Theta_p|_{A \times YZ \times \{0\}}$ factors through the inclusion $YZ \subseteq AYZ$ and that $\Theta_p|_{t=1} \stackrel{\text{def}}{=} \Theta_p|_{A \times YZ \times \{1\}}$ factors through the inclusion $AZ \subseteq AYZ$. By applying $\text{map}^G(-, XYZ)$ to these square homotopies we obtain the maps

$$(6.10) \quad (\Xi_2)_p: \mathcal{H}_2 \xrightarrow{(\Theta_p|_{t=0})^* \times (\Theta_p|_{t=1})^*} \mathcal{D}_2 \quad \text{and} \quad (\Xi_0)_p: \mathcal{H}_2 \xrightarrow{(\Theta_p|_{t=0})^* \times (\Theta_p|_{t=1})^*} \mathcal{D}_0.$$

Naturality of the construction of Θ with respect to the inclusion $X' \subseteq X$ implies the commutativity of the squares

$$\begin{array}{ccc} \mathcal{H}_3 & \xrightarrow{h_{31}} & \mathcal{H}_1 \\ (\Xi_3)_p \downarrow & & \downarrow (\Xi_1)_p \\ \mathcal{D}_3 & \xrightarrow{d_{31}} & \mathcal{D}_1 \end{array} \quad \begin{array}{ccc} \mathcal{H}_2 & \xrightarrow{h_{20}} & \mathcal{H}_0 \\ (\Xi_2)_p \downarrow & & \downarrow (\Xi_0)_p \\ \mathcal{D}_2 & \xrightarrow{d_{20}} & \mathcal{D}_0. \end{array}$$

Furthermore, by applying $\text{map}^G(-, XYZ)$ to the commutative square

$$(6.11) \quad \begin{array}{ccc} (A \times YZ) \sqcup (A \times YZ) & \xrightarrow{\text{incl}_0 + \text{incl}_1} & A \times YZ \times I \\ \Theta_p|_{t=0} \sqcup \Theta_p|_{t=1} \downarrow & & \downarrow \Theta_p \\ YZ \sqcup AZ & \xrightarrow{i_{YZ} + i_{AZ}} & AYZ. \end{array}$$

we obtain the commutativity of

$$\begin{array}{ccc} \mathcal{H}_3 & \xrightarrow{h_{32}} & \mathcal{H}_2 \\ (\Xi_3)_p \downarrow & & \downarrow (\Xi_2)_p \\ \mathcal{D}_3 & \xrightarrow{d_{32}} & \mathcal{D}_2 \end{array} \quad \begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{h_{10}} & \mathcal{H}_0 \\ (\Xi_1)_p \downarrow & & \downarrow (\Xi_0)_p \\ \mathcal{D}_1 & \xrightarrow{d_{10}} & \mathcal{D}_0 \end{array}$$

Therefore Ξ_0, \dots, Ξ_3 give rise to a homotopy $\Xi: I \times \mathcal{H} \rightarrow \mathcal{D}$ in \mathfrak{S} . Composition with $\Upsilon: \mathcal{A} \rightarrow \mathcal{H}$ gives a homotopy $\Xi \circ \Upsilon: I \times \mathcal{A} \rightarrow \mathcal{D}$ parametrized by $p \in I$.

It remains to show that $(\Xi \circ \Upsilon)|_{p=0} = \Psi \circ \Pi$ and that $(\Xi \circ \Upsilon)|_{p=1} = \Pi \circ \Phi$. We start with $p = 1$. By Proposition 4.8, $\Theta|_{p=1} = \beta$ is the natural map $A \times YZ \times I \xrightarrow{\pi} A(YZ) \cong AYZ$ in

(4.2). By the definition of Φ_1, Φ_3 in (6.4) and Π_1, Π_3 in (6.5) and Υ_1, Υ_3 in (6.8) and Ξ_1, Ξ_3 in (6.9), it follows that for $i = 1, 3$

$$(\Xi_i)|_{p=1} \circ \Upsilon_i = \beta^* \circ J_Z = \pi^* \circ J_Z = \Pi_i \circ \Phi_i.$$

Proposition 4.8 shows that $\Theta_{p=1}|_{t=0} = \beta|_{A \times YZ \times \{0\}}$ is the projection $A \times YZ \rightarrow YZ$ and that $\Theta_{p=1}|_{t=1} = \beta|_{A \times YZ \times \{1\}}$ is the composition of the projection $A \times YZ \rightarrow A$ followed by the inclusion into AZ . Since $(\Xi_0)_p$ and $(\Xi_2)_p$ are obtained by applying $\text{map}^G(-, XYZ)$ to the first column of (6.11), the commutative triangle in (4.6) together with (6.4) and (6.5) show that for $i = 0, 2$

$$(\Xi_i)_{p=1} \circ \Upsilon_i = (\pi_{YZ}^* \circ J_Z) \times (\pi_A^* \circ (i_A^{AZ})^* \circ J_Z) = (\pi_{YZ}^* \circ J_Z) \times (\pi_A^* \circ \text{incl}_*) = \Pi_i \circ \Phi_i.$$

It follows that

$$(\Xi \circ \Upsilon)|_{p=1} = \Pi \circ \Phi.$$

It remains to show that $(\Xi \circ \Upsilon)|_{p=0} = \Psi \circ \Pi$. First, we claim that for $i = 1, 3$

$$(\Xi_i)_{p=0} \circ \Upsilon_i = \alpha^* \circ J_Z = \psi_{A \times I, Y, XY, Z} \circ \pi^* = \Psi_i \circ \Pi_i.$$

The first equality follows from the definitions of Ξ_i and Υ_i in (6.9) and (6.8), where $i = 1, 3$, and from Definition 4.7; The second equality follows from the commutative square (4.8) in Proposition 4.10, and the third from the definitions of Ψ_i and Π_i in (6.7) and (6.6).

Let $i = 0, 2$. We claim that

$$\begin{aligned} (\Xi_i)_{p=0} \circ \Upsilon_i &= (\alpha|_{A \times YZ \times \{0\}})^* \times (\alpha|_{A \times YZ \times \{1\}})^* \circ (J_Z \times J_Z) = \\ &= (\psi_{A, Y, XY, Z} \circ (\pi_{A \times Y}^Y)^*) \times (\psi_{A, Y, XY, Z} \circ (\pi_{A \times Y}^A)^*) = \Psi_i \circ \Pi_i. \end{aligned}$$

The first equality follows from the definition of Υ_i and Ξ_i in (6.8) and (6.9); The second follows from (4.9) and (4.10) in Proposition 4.10, and the third from the definition of Ψ_i and Π_i in (6.7) and (6.6). It follows that $(\Xi \circ \Upsilon)|_{p=0} = \Psi \circ \Pi$. Q.E.D

This completes step (d) of the proof. We are now ready to complete the proof of the proposition as outlined in step (e).

Claims 3 and 4 and the functoriality of Λ imply that $\Lambda(\mathcal{A}, \underline{x}) \xrightarrow{\Lambda(\Psi \circ \Pi)} \Lambda(\mathcal{D}, \Psi \circ \Pi(\underline{x}))$ is a k -equivalence for any choice of basepoint \underline{x} in \mathcal{A} . Claim 5 together with Lemma 3.4 show that $\Lambda(\mathcal{A}, \underline{x}) \xrightarrow{\Lambda(\Pi \circ \Phi)} \Lambda(\mathcal{D}, \Pi \Phi(\underline{x}))$ is a k -equivalence. From Claim 3 and the functoriality of Λ we deduce that $\Lambda(\mathcal{A}, \underline{x}) \xrightarrow{\Lambda(\Phi)} \Lambda(\mathcal{B}, \Phi(\underline{x}))$ is a k -equivalence for any basepoint \underline{x} in \mathcal{A} .

By hypothesis (3), $\text{conn}(XY) \geq \dim X + k + 1$ and therefore also $\text{conn}(XYZ) \geq \dim X + k + 1$, see for example [7, Lemma 2.3]. Thanks to hypothesis (1), we may apply Corollary 2.5 to $X' \subseteq X$ and to XY and XYZ and deduce, in light of the definitions of \mathcal{A} and \mathcal{B} , that the horizontal arrows in the following square are $(k+1)$ -equivalences.

$$\begin{array}{ccc} \mathcal{A}_2 & \xrightarrow{a_{20}} & \mathcal{A}_0 \\ \Phi_2 \downarrow & & \downarrow \Phi_0 \\ \mathcal{B}_2 & \xrightarrow{b_{20}} & \mathcal{B}_0 \end{array}$$

In particular, their fibres are $(k+1)$ -connected, and therefore Φ_2 and Φ_0 induce k -equivalences among them. We have already seen that $\Lambda(\mathcal{A}, \underline{x}) \xrightarrow{\Lambda(\Phi)} \Lambda(\mathcal{B}, \Phi(\underline{x}))$ are k -equivalences, therefore we may apply Lemma 3.2(3) to deduce that in the commutative square

$$\begin{array}{ccc} \mathcal{A}_3 & \xrightarrow{a_{31}} & \mathcal{A}_1 \\ \Phi_3 \downarrow & & \downarrow \Phi_1 \\ \mathcal{B}_3 & \xrightarrow{b_{31}} & \mathcal{B}_1 \end{array}$$

the vertical arrows induce k -equivalences on all the fibres of a_{31} and b_{31} . Given the definition of Φ_1 and Φ_3 in (6.4), this is exactly the claim of this proposition. \square

7. THE LIMIT GROUPS

The purpose of this section is to prove Proposition 7.3 below. It will make an essential use of equivariant stable homotopy theory. Our main reference is Lewis-May-Steinberg [5]. Let G be a finite group. We will fix a “universe” \mathcal{U} for the representations of G and an indexing set \mathcal{A} for \mathcal{U} [5, Definition 2.1]. A spectrum E is a collection of pointed G -spaces $E(V)$ for every $V \subseteq \mathcal{A}$, subject to some conditions [5, Def. 2.1]. There is a functor Σ^∞ from the category of pointed G -spaces to the category of G -spectra. It has a right adjoint Ω^∞ which assigns to a spectrum E its zeroth space [5, Proposition 2.3]. We denote, as usual, $Q = \Omega^\infty \circ \Sigma^\infty$ [5, p. 14]. If A is a pointed G -space, the function spectrum $F(A, E)$ is obtained by applying $F(A, -)$ to each space of E [5, Def. 3.2]. If $H \leq G$, there is a fixed point spectrum E^H which is obtained, once again, by taking H -fixed points of the spaces of E [5, Def. 3.7]. It has a structure of a WH -spectrum where $WH = N_G H/H$.

The Borel construction of a G -space X is the orbit space $EG \times_G X = (EG \times X)/G$. Let X be a pointed G -space. Denote $EG_+ \wedge_G X \stackrel{\text{def}}{=} (EG_+ \wedge X)/G$. The following important result [5, Section V.11], attributed to tom-Dieck, gives a complete description of the fixed point spectrum $(\Sigma^\infty X)^G$

$$(\Sigma^\infty X)^G \simeq \bigvee_H \Sigma^\infty \left(EWH_+ \wedge_{WH} X^H \right)$$

where H runs through representatives of the conjugacy classes of the subgroups of G and where $WH = N_G H/H$ acts in the natural way on X^H .

Let U be a real representation of G . Let S^U denote the one point compactification of U with basepoint $\alpha_1 = \infty$. Clearly $\alpha_0 = 0 \in U$ is also fixed by G . Notice that for any $H \leq G$ we have $(S^U)^H = S^V$ where $V = U^H$. It has a natural action of WH .

Let $C(f)$ denote the mapping cone of a map $f: A \rightarrow B$ of unpointed G -spaces; it is equipped with a natural basepoint (the “tip of the cone”). If $A \subseteq B$ we write $C(B, A)$ for the mapping cone of the inclusion.

For a G -space X with fixed points x_0, x_1 , denote by $\mathbb{P}_{x_0, x_1} X$ the space of paths $\omega: I \rightarrow X$ with $\omega(0) = x_0$ and $\omega(1) = x_1$. It has a natural action of G .

Lemma 7.1. (1) *Let X be a pointed finite CW-complex such that $\tilde{H}_i(X) = 0$ for all $0 \leq i \leq m$. Then $\pi_i \Sigma^\infty X = 0$ for all $0 \leq i \leq m$.*
 (2) *Let X be a finite G -CW complex such that $\tilde{H}_i(X) = 0$ for all $0 \leq i \leq m$. Then $\pi_i \Sigma^\infty (EG_+ \wedge_G X) = 0$ for all $0 \leq i \leq m$.*

Proof. (1). This follows from Atiyah-Hirzebruch spectral sequence $\tilde{H}_i(X, E_j(*)) \Rightarrow \tilde{E}_{i+j}(X)$ applied to the sphere spectrum $E = \mathbf{S}$.

(2). There is a G -cofibre sequence where EG retracts off $EG \times X$ equivariantly (via the basepoint of X).

$$EG \xleftarrow{\quad} EG \times X \longrightarrow EG_+ \wedge X$$

By taking G -orbits we get a cofibre sequence

$$BG \xleftarrow{\quad} X_{hG} \longrightarrow EG_+ \wedge_G X$$

with BG retracting off $X_{hG} = EG \times_G X$. The Serre spectral sequence $H_i(BG, H_j(X)) \Rightarrow H_{i+j}(X_{hG})$ of the fibration $X_{hG} \rightarrow BG$ shows that $H_i(BG) \rightarrow H_i(X_{hG})$ is an isomorphism for all $0 \leq i \leq m$ and therefore $\tilde{H}_i(EG_+ \wedge_G X) = 0$ for all $0 \leq i \leq m$. The result follows from item (1). \square

Lemma 7.2. *Let V be a G -representation and $X \subseteq V$ a finite G -CW complex. Set $n = \dim V$. Then there are isomorphisms for all $0 \leq i \leq n - 2$*

$$\pi_{i+1}\Sigma^\infty(EG_+ \wedge_G C(S^V, S^V \setminus X)) \cong \pi_i\Sigma^\infty(EG_+ \wedge_G (S^V \setminus X))$$

Proof. Lemma 7.1(2) shows that $\pi_i\Sigma^\infty(EG_+ \wedge_G S^V) = 0$ for $0 \leq i \leq n - 1$. The long exact sequence in stable homotopy groups of the cofibration $EG_+ \wedge_G (S^V \setminus X) \rightarrow EG_+ \wedge_G S^V \rightarrow EG_+ \wedge_G C(S^V, S^V \setminus X)$ gives the result. \square

Proposition 7.3. *Let U be a representation of G and let $X \subseteq U$ be a finite G -CW complex. Let $k \geq 0$. Assume that for any $H \in \text{Iso}_G(X)$*

- (a) $\dim X^H < \dim U^H$.
- (b) $\dim X^H - \dim(\bigcup_{K \geq H} X^K) > k + 1$.
- (c) $(S^U \setminus X)^H$ is WH -equivariantly homotopy equivalent to a WH -CW complex.

Then $\text{map}^G(X, S^U)$ is path connected and for all $1 \leq i \leq k + 1$

$$(7.1) \quad \pi_i \text{map}^G(X, S^U) \cong \bigoplus_{(H) \subseteq \text{Iso}_G(X)} \pi_i \Sigma^\infty \left(EWH_+ \wedge_{WH} C(S^U, S^U \setminus X)^H \right).$$

If in addition

- (d) there exists a G -map $X \xrightarrow{\eta} \mathbb{P}_{\alpha_0, \alpha_1} S^U$ such that $X^H \xrightarrow{\eta} \mathbb{P}_{\alpha_0, \alpha_1} (S^U)^H$ is a $(\dim X^H + k)$ -equivalence for any $H \in \text{Iso}_G(X)$

then for any basepoint $f \in \text{map}^G(X, X)$ and every $0 \leq i \leq k$ there are isomorphisms (bijection for $i = 0$):

$$(7.2) \quad \pi_i \text{map}^G(X, X) \cong \bigoplus_{(H) \subseteq \text{Iso}_G(X)} \pi_i \Sigma^\infty \left(EWH_+ \wedge_{WH} (S^U \setminus X)^H \right).$$

Proof. We will prove by induction on the filtration $\{X_q\}_{q=0}^r$ of X in (5.2) that $\text{map}^G(X_q, S^U)$ is path connected and that there are isomorphisms for all $1 \leq i \leq k + 1$

$$(7.3) \quad \pi_i \text{map}^G(X_q, S^U) \cong \bigoplus_{(H) \subseteq \text{Iso}_G(X_q)} \pi_i \Sigma^\infty \left(EWH_+ \wedge_{WH} C(S^U, S^U \setminus X)^H \right).$$

The base of induction is a triviality since $X_0 = \emptyset$. Assume that (7.3) holds for $q - 1$ and we prove it for $1 \leq q \leq r$. If $H_q \notin \text{Iso}_G(X)$ then $X_q = X_{q-1}$ and the induction step is trivial. So we assume that $H = H_q$ is in $\text{Iso}_G(X)$.

Choose some basepoint $f \in \text{map}^G(X_q, S^U)$. We obtain a fibre sequence (over $f|_{X_{q-1}}$)

$$F \rightarrow \text{map}^G(X_q, S^U) \xrightarrow{j^*} \text{map}^G(X_{q-1}, S^U).$$

The hypotheses imply that

$$\dim_H(X_q, X_{q-1}) \leq \dim X_q^H = \dim X^H \leq \dim U^H - 1 = \text{conn}(S^U)^H.$$

We can apply Corollary 2.5 (with $Y = S^U$ and $k = 0$) to deduce that j^* is bijective on components and that $\pi_0 F = *$. In particular it follows that $\text{map}^G(X_q, S^U)$ is path connected, as needed, and therefore we may assume, in connection to (7.3), that the basepoint f is the null map.

Since $\text{Iso}_G(X_q) = \text{Iso}_G(X_{q-1}) \cup (H)$, in order to complete the induction step for (7.3) it remains to show that for every $1 \leq i \leq k + 1$

- (i) $\pi_i F \rightarrow \pi_i \text{map}^G(X_q, S^U)$ is split injective, and
- (ii) $\pi_i F \cong \pi_i \Sigma^\infty(EWH_+ \wedge_{WH} C(S^U, S^U \setminus X)^H)$.

For the rest of the proof set $V = U^H$ and $n = \dim V$. Proposition 5.3 yields the following morphism of fibrations which induces a homeomorphism on the fibres (over the null maps)

$$\begin{array}{ccccc} F & \longrightarrow & \text{map}^G(X_q, S^U) & \xrightarrow{j^*} & \text{map}^G(X_{q-1}, S^U) \\ \cong \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & \text{map}^{WH}(X^H, S^V) & \xrightarrow{j^*} & \text{map}^{WH}(X_{q-1}^H, S^V). \end{array}$$

Therefore, we will be finished if we prove (ii) and that $\pi_i F \rightarrow \pi_i \text{map}^{WH}(X_{q-1}^H, S^V)$ is split injective for all $1 \leq i \leq k+1$.

Since $QS^V = \text{colim}_{V'} \Omega^{V'} \Sigma^{V'} S^V$ [5, p. 14], Freudenthal's theorem implies that the natural map $S^V \rightarrow QS^V$ is a non-equivariant $(2n-2)$ -equivalence. Hypotheses (a) and (b) imply that

$$\dim X^H \geq k+2 + \dim \left(\bigcup_{K \geq H} X^K \right) \geq k+1$$

because $\dim(\cup_{K \geq H} X^K) \geq -1$. In particular

$$\dim(X^H, X_{q-1}^H) + k+1 \leq \dim X^H + k+1 \leq 2 \cdot \dim X^H \leq 2(n-1) = 2n-2.$$

Application of Proposition 5.2(2) and Lemma 2.4 to $X_{q-1}^H \subseteq X^H$ and $S^V \rightarrow QS^V$ shows that in the commutative diagram

$$(7.4) \quad \begin{array}{ccccc} F & \longrightarrow & \text{map}^{WH}(X^H, S^V) & \xrightarrow{j^*} & \text{map}^{WH}(X_{q-1}^H, S^V) \\ \downarrow & & \downarrow & & \downarrow \\ F' & \longrightarrow & \text{map}^{WH}(X^H, QS^V) & \xrightarrow{\ell^*} & \text{map}^{WH}(X_{q-1}^H, QS^V). \end{array}$$

the map $F \rightarrow F'$ between the fibres (over the null maps) is a $(k+1)$ -equivalence. Thus, to complete the induction step of (7.3) it suffices to prove that for every $1 \leq i \leq k+1$

- (i') $\pi_i F' \rightarrow \pi_i \text{map}^{WH}(X^H, QS^V)$ is split injective, and
- (ii') $\pi_i F'$ is isomorphic to the groups in (ii).

Let A be a pointed G -CW complex. The definitions of Ω^∞ and $Q = \Omega^\infty \Sigma^\infty$ [5, p. 14], of the function spectra $F(A, E)$ [5, Prop. 3.6], and of fixed point spectra [5, Def. 3.7], imply that there are natural homeomorphisms

$$\begin{aligned} \text{map}^{WH}(A, QS^V) &= F(A_+, \Omega^\infty \Sigma^\infty S^V)^{WH} \cong (\Omega^\infty F(A_+, \Sigma^\infty S^V))^{WH} \\ &= \Omega^\infty (F(A_+, \Sigma^\infty S^V)^{WH}). \end{aligned}$$

Therefore the fibration ℓ^* in the 2nd row of (7.4) is obtained by applying the functor Ω^∞ and WH -fixed points to the morphism of WH -spectra

$$F(X_+^H, \Sigma^\infty S^V) \xrightarrow{\ell^*} F((X_{q-1}^H)_+, \Sigma^\infty S^V).$$

We will now exploit V -duality [5, Chap. III]. Recall that $C(X, \emptyset) = X_+$ where X is an unpointed space [5, page 142]. The definition of V -duality [5, Defn. 3.4] together with the formula for the map $\epsilon/(?)$ [5, Prop. 3.1] and the construction of V -duality for compact G -ENRs [5, Construction 4.5, page 145], give rise to the following homotopy commutative diagram of WH -spectra

$$\begin{array}{ccc} F(X_+^H, \Sigma^\infty S^V) & \xrightarrow{\ell^*} & F((X_{q-1}^H)_+, \Sigma^\infty S^V) \\ \epsilon^\#(X^H) \uparrow & & \uparrow \epsilon^\#(X_{q-1}^H) \\ \Sigma^\infty C(S^V, S^V \setminus X^H) & \xrightarrow{\ell^*} & \Sigma^\infty C(S^V, S^V \setminus X_{q-1}^H). \end{array}$$

By definition of V -duality (or by construction), the vertical arrows in this diagram are weak equivalences of WH -spectra [5, Def. 4.4]. By applying the fixed points functor $(-)^{WH}$, and the functor Ω^∞ we see that in order to prove (i') and (ii') it suffices to prove that

(i'') $\pi_i(\Sigma^\infty C(S^V, S^V \setminus X^H))^{WH} \xrightarrow{\pi_i(\ell^*)} \pi_i(\Sigma^\infty C(S^V, S^V \setminus X_{q-1}^H))^{WH}$ is split surjective for all $1 \leq i \leq k+1$ and surjective for $i = k+2$, and

(ii'') the kernels of the homomorphisms in (i'') are isomorphic to the groups in (ii) for all $1 \leq i \leq k+1$.

On the level of spectra, [5, Section V.11] quoted above shows that the map in (i'') is induced by the map of spectra

$$\bigvee_{(K)} \Sigma^\infty(EWK_+ \wedge_{WK} C(S^V, S^V \setminus X^H)^K) \rightarrow \bigvee_{(K)} \Sigma^\infty(EWK_+ \wedge_{WK} C(S^V, S^V \setminus X_{q-1}^H)^K)$$

where the sum is over all the conjugacy classes of subgroups $K \leq WH$ and by WK we mean $N_{WH}(K)/K$. Any $K \leq WH$ has the form $K = L/H$ for some $H \leq L \leq N_G H$. If $K \neq 1$ then $L \supsetneq H$ and in this case $X^L = X_{q-1}^L$ and it follows that the maps of the summands corresponding to $K \neq 1$ are equivalences. It remains to examine the summand $K = 1$, namely the map

$$(7.5) \quad \Sigma^\infty(EWH_+ \wedge_{WH} C(S^V, S^V \setminus X^H)) \rightarrow \Sigma^\infty(EWH_+ \wedge_{WH} C(S^V, S^V \setminus X_{q-1}^H)).$$

The hypotheses and Proposition 5.1(3) show that

$$\dim X_{q-1}^H \leq \dim X^H - k - 2 \leq n - k - 3.$$

In particular $H^i(X_{q-1}^H) = 0$ for all $i \geq n - k - 2$. Alexander duality implies that $\tilde{H}_i(S^V \setminus X_{q-1}^H) = 0$ for all $0 \leq i \leq k+1$. Also, $\tilde{H}_i(S^V) = 0$ for all $0 \leq i \leq k+1$ since $\text{conn}(S^V) = n - 1 \geq \dim X^H \geq k + 1$. There is a cofibre sequence

$$EWH_+ \wedge_{WH} (S^V \setminus X_{q-1}^H) \xrightarrow{\gamma} EWH_+ \wedge_{WH} S^V \xrightarrow{\gamma} EWH_+ \wedge_{WH} C(S^V, S^V \setminus X_{q-1}^H).$$

The long exact sequence in stable homotopy groups together with Lemma 7.1(2) show that π_i of the right hand side of (7.5) vanishes for $0 \leq i \leq k+1$ and that $\pi_{k+2} \Sigma^\infty \gamma$ is surjective. Also π_{k+2} of (7.5) is surjective because $\pi_{k+2} \Sigma^\infty \gamma$ factors through it. In particular (i'') and (ii'') follow and the induction step is complete.

Let $X \xrightarrow{\eta} \mathbb{P}_{\alpha_0, \alpha_1} S^U$ be as in hypothesis (d). Applying Lemma 2.4 with $\emptyset \subseteq X$ and with η shows that $\text{map}^G(X, X) \rightarrow \text{map}^G(X, \mathbb{P}_{\alpha_0, \alpha_1} S^U)$ is a k -equivalence. By inspection, and since we have shown that $\text{map}^G(X, S^U)$ is path connected,

$$\text{map}^G(X, \mathbb{P}_{\alpha_0, \alpha_1} S^U) \cong \mathbb{P}_{\alpha_0, \alpha_1} \text{map}^G(X, S^U) \simeq \Omega \text{map}^G(X, S^U).$$

We have seen that if $H \in \text{Iso}_G(X)$ and $n = \dim U^H$ then $n - 1 \geq \dim X^H \geq k + 1$. Lemmas 7.2 and 7.1(2) apply to show that for $0 \leq i \leq k$ there are isomorphisms (bijection if $i = 0$)

$$\begin{aligned} \pi_i \text{map}^G(X, X) &\cong \pi_{i+1} \text{map}^G(X, S^U) \cong \bigoplus_{(H) \subseteq \text{Iso}_G(X)} \pi_{i+1} \Sigma^\infty (EWH_+ \wedge_{WH} C(S^U, S^U \setminus X)^H) \\ &\cong \bigoplus_{(H) \subseteq \text{Iso}_G(X)} \pi_i \Sigma^\infty (EWH_+ \wedge_{WH} (S^U \setminus X)^H) \end{aligned}$$

□

REFERENCES

- [1] J. C. Becker and R. E. Schultz. Spaces of equivariant self-equivalences of spheres. *Bull. Amer. Math. Soc.*, 79:158–162, 1973.
- [2] Wojciech Chachólski and Jérôme Scherer. Homotopy theory of diagrams. *Mem. Amer. Math. Soc.*, 155(736):x+90, 2002.
- [3] H. Hauschild. Äquivariante Homotopie. I. *Arch. Math. (Basel)*, 29(2):158–165, 1977.

- [4] Sören Illman. Smooth equivariant triangulations of G -manifolds for G a finite group. *Math. Ann.*, 233(3):199–220, 1978.
- [5] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.
- [6] Assaf Libman. On the homotopy groups of the self-equivalences of linear spheres. *Proc. Edinb. Math. Soc. (2)*, 59(2):445–462, 2016.
- [7] John Milnor. Construction of universal bundles. II. *Ann. of Math. (2)*, 63:430–436, 1956.
- [8] Reinhard Schultz. Homotopy decompositions of equivariant function spaces. II. *Math. Z.*, 132:69–90, 1973.
- [9] N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:133–152, 1967.
- [10] N. P. Strickland. The category of cgwh spaces. Available from <https://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf>, 2009.
- [11] Tammo tom Dieck. *Transformation groups*, volume 8 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1987.
- [12] George W. Whitehead. *Elements of homotopy theory*, volume 61 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF ABERDEEN, FRASER NOBLE BUILDING, ABERDEEN AB24 3UE, U.K.

E-mail address: `a.libman@abdn.ac.uk`